Scaling solution, radion stabilization, and initial condition for brane-world cosmology

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(Received 22 October 2003; published 25 March 2004)

We propose a self-consistent and dynamical scenario which gives rise to well-defined initial conditions for five-dimensional brane-world cosmologies with radion stabilization. At high energies, the five-dimensional effective theory is assumed to have a scale invariance so that it admits an expanding scaling solution as a future attractor. The system automatically approaches the scaling solution and, hence, the initial condition for the subsequent low-energy brane cosmology is set by the scaling solution. At low energies, the scale invariance is broken and a radion stabilization mechanism drives the dynamics of the brane-world system. We present an exact, analytic scaling solution for a class of scale-invariant effective theories of five-dimensional brane-world models which includes the five-dimensional reduction of the Horava-Witten theory, and provide convincing evidence that the scaling solution is a future attractor.

DOI: 10.1103/PhysRevD.69.064029 PACS number(s): 04.50.+h, 98.80.Cq

I. INTRODUCTION

Phenomenological models in which our Universe is a four-dimensional (4D) hypersurface, or a 3-brane, embedded in a higher-dimensional spacetime is currently of great interest. In the brane-world scenario, ordinary matter fields are confined to the brane, while the gravitational field can propagate in the extra dimensions (i.e., in the "bulk") [1–5]. In particular, Randall and Sundrum (RS2) proposed a self-consistent brane-world scenario with 5D anti–de Sitter (AdS) bulk spacetime and showed that 4D gravity is also localized on the brane if the brane tension is positive and fine-tuned [5]. Many aspects of the Randall-Sundrum scenarios have been investigated, including weak gravity [6–8], the effective Einstein equation [9,10], cosmology [11–18], black holes [19–23] and so on.

In the brane-world scenario, the evolution of our Universe can be considered in two ways. On the one hand, it is the expansion of the induced geometry on our brane. Note that ordinary matter fields are assumed to be confined on the brane and, thus, propagate in the induced geometry. This picture is not very different from what we are accustomed to in the standard 4D cosmology. Indeed, the generalized Friedmann equation for the homogeneous, isotropic brane universe in the RS2 scenario reduces to the standard Friedmann equation at low energy with corrections [11–15]. In this picture the bulk geometry also evolves,[1] corresponding to the expansion of our Universe. On the other hand, the same evolution of our universe can be seen in a different way as a brane motion in a higher-dimensional spacetime. In this picture the generalized Friedmann equation is understood as the equation of motion for the brane moving in the bulk spacetime. In the RS2 scenario this picture is remarkably simple for the homogeneous, isotropic brane universe. Indeed, the bulk spacetime is always locally the AdS-Schwarzschild spacetime, irrespective of the brane motion [16–18]. In other words, the brane motion does not generate waves in the bulk. Unfortunately, in the more realistic scenarios which we shall consider in this paper this remarkable property is not satisfied. Nonetheless, it is still useful to keep in mind that there are two different ways to describe the same evolution of the brane-world cosmology: the expansion of the induced geometry and the brane motion in the bulk.

Prior to the RS2 scenario, Randall-Sundrum had proposed a similar but different scenario called the RS1 scenario [4]. This scenario involves two branes with fine-tuned tensions, one positive and another negative, and aims to solve the gauge hierarchy problem. In order to obtain an appropriate hierarchy between the Planck scale and the electroweak scale, the distance between the two branes must be set to about 50 times the bulk curvature scale. This itself is not a severe fine-tuning nor a problem at all, but it would be more satisfactory if this value could be dynamically realized. In the RS1 scenario, unfortunately, from the 4D brane viewpoint the interbrane distance represents a massless scalar degree of freedom and, thus, is arbitrary. Moreover, the existence of this massless scalar mode, called the *radion*, causes a more serious problem: the low energy 4D gravity on our brane is not general relativity but a Brans-Dicke type theory with a too small Brans-Dicke parameter to be consistent with gravitational experiments [6]. Here, the radion plays the role of the Brans-Dicke scalar.

The problem of too large a deviation from the Einstein theory can be fixed if the radion obtains a nonzero mass. With a nonvanishing radion mass, the radion is not excited and becomes irrelevant at energies much lower than the radion mass, so that the 4D Einstein gravity is restored at low energy. This is called *radion stabilization*. Goldberger and Wise proposed a mechanism to stabilize the radion by introducing a scalar field in the bulk [24]. This scalar field has a potential in the bulk (bulk potential) and couples to branes via its potentials localized on the branes (brane potentials).

[1]Here the bulk geometry is supposed to be seen in the Gaussian normal coordinate based on the world-volume of our brane.

The introduction of scalar field(s) in the bulk is, of course, favorable from the viewpoint of string theory since a compactification from 10 or 11 dimensions down to 5 dimensions in general introduces many 5D scalar fields. In this sense, the generalized RS1 scenarios with radion stabilization seem more realistic than the RS2 model. With the Goldberger-Wise mechanism, 4D Einstein gravity is indeed shown to be recovered on our brane [25–29].[2] The radion stabilization also helps recovery of the standard Friedmann equation at low energy [31,32].

The introduction of the bulk scalar complicates the brane-world scenario in (at least) two different ways. First, the gauge hierarchy problem is now entangled with the cosmological constant problem. The essential reason for this is that both the 4D cosmological constant and the stabilized value of the interbrane distance depend on the bulk potential and the brane potentials of the scalar field in a nontrivial way.

Second, the dynamics of the system becomes much more complicated. The bulk geometry is no longer simple if a brane is moving; i.e., if our 4D universe is expanding. A general brane motion generates waves of the bulk scalar field and makes the bulk geometry very complicated. This makes it extremely difficult to analyze the general dynamics of brane-world cosmology analytically. Recent development of BRANECODE [33] has made it possible to analyze the dynamics numerically, but there remains the problem of initial conditions. Namely, the system is so rich that it is not *a priori* trivial to choose physically relevant initial conditions for the numerical study.

The purpose of this paper is to shed light on this second point in a particular way. In the rest of this paper, our main concern will be the dynamics of the brane-world system with radion stabilization. In particular, we shall propose a new dynamical scenario to give an initial condition for the brane-world cosmology with radion stabilization.

II. PHYSICAL SCENARIO

Suppose that the 5D theory has an approximate scale invariance at relatively high energy (but still much lower than the Planck scale) so that the 5D effective action allows a scaling solution. For example, we might imagine that the bulk potential $V(\phi)$ and the brane potentials $\lambda_{\pm}(\phi_{\pm})$ have the form

$$V(\phi)=V_0 e^{-2\alpha\kappa_5\phi}, \quad \lambda_{\pm}(\phi_{\pm})=\lambda_0^{\pm}[e^{-\alpha\kappa_5\phi_{\pm}}+e^{\beta_{\pm}\kappa_5\phi_{\pm}}], \tag{1}$$

where $\phi_{\pm}$ is the pullback of ϕ on each brane and $\alpha+\beta_{\pm}\neq 0$. We assume that $-(\alpha+\beta_{\pm})\kappa_5\phi_{\pm}\gg 1$ initially so that the term $e^{\beta_{\pm}\kappa_5\phi_{\pm}}$ can be neglected and the system has the scaling invariance.

It is worth mentioning here that potentials of the form (1) arise in many theories of the fundamental interactions. For example, compactification of higher-dimensional theories, including superstring theories, gives rise to exponential terms in the effective potential. Terms with different exponents come from different physical effects, e.g., higher-dimensional cosmological constant, curvature of compact manifold, antisymmetric field flux, Casimir effects, etc. We shall provide more motivations for exponential potentials in Sec. IV A.

Since the interbrane distance has the dimension of length, we expect that it should be proportional to time and, thus, expanding for the scaling solution. In this paper, we shall find such a scaling solution and present convincing evidence that it is a future attractor. For the moment, we just assume that the scaling solution is a future attractor. In this case, the system should approach it dynamically. Subsequently, since the Hubble expansion rate decays as $1/t$ and the size of the extra dimension increases as t for the scaling solution, the system should enter a lower and lower energy regime. Hence, the scale invariance is no more than an approximate symmetry. In the above example (1), when $-(\alpha+\beta_{\pm})\kappa_5\phi_{\pm}$ becomes of order unity, the term $e^{\beta_{\pm}\kappa_5\phi_{\pm}}$ in the brane potentials cannot be neglected any more and the approximate scaling invariance will be broken.

Depending on the form of the effective action relevant to the lower energy regime, the interbrane distance, called the radion, may play the role of an inflaton. Hence, we will have a 4D inflation on our brane driven by the radion. When the radion-induced inflation ends, the radion is stabilized and the fields confined on our brane can be reheated due to the oscillatory behavior of the system around the stabilized configuration. Subsequently, the conventional standard cosmology is realized. Since the radion is eventually stabilized, Einstein weak gravity is restored at low energy [25,27–29].

Hence, in this hypothetical scenario, the role of the scaling solution is to provide the initial condition for the radion-induced inflation and the subsequent evolution. To our knowledge there has been no well-defined scenario to provide an initial condition for the brane-world cosmology, except perhaps for the creation-from-nothing scenario [34,35]. If the scaling solution is a future attractor then the classical dynamics will automatically drive the system to the well-defined initial condition for the following evolution of the brane-world cosmology.

In the remainder of this paper we shall find an exact, analytic scaling solution and present convincing evidence for its attractor behavior.

III. BASIC EQUATIONS

In this section we present the basic equations for general potentials. In the following sections we apply these equations to a specific model with exponential potentials, motivated by string-inspired phenomenology.

In the 5D bulk we consider Einstein gravity and a scalar field:

[2] For an apparent conflict with the picture in Ref. [30], and its resolution, see Ref. [28].

$$I_5=\int dx^5\sqrt{-g}\left[\frac{R}{2\kappa_5^2}-\frac{1}{2}\partial^M\phi\partial_M\phi-V(\phi)\right]. \tag{2}$$

Hence, the 5D Einstein equation is

$$G_{MN}=\kappa^2T_{MN},$$

$$T_{MN}=\partial_M\phi\partial_N\phi-\left[\frac{1}{2}\partial^L\phi\partial_L\phi+V(\phi)\right]g_{MN}. \tag{3}$$

The field equation for ϕ,

$$\nabla^2\phi-V'(\phi)=0, \tag{4}$$

where ∇ is the covariant derivative compatible with the metric g_{MN}, follows automatically from the Einstein equation because of the Bianch identity $\nabla^\mu G_{\mu\nu}=0$.

We assume that the extra dimension has the topology of Z_2/S^1 and consider two end-of-the-world branes on the two fixed points. The tension $\lambda_\pm$ of each brane in general depends on the pullback $\phi_\pm$ of the scalar field ϕ. Hence, the brane action is of the form

$$I_4=-\int dx_-^4\sqrt{-q_-}\lambda_-(\phi_-)-\int dx_+^4\sqrt{-q_+}\lambda_+(\phi_+), \tag{5}$$

where $x_\pm$ represents 4D coordinates on each brane and $q_\pm$ is the determinant of the induced metric.

Note that we need to add the Gibbons-Hawking boundary term to the action, depending on the precise definition of I_5. If I_5 includes the integration over the thin layers corresponding to the branes then the Gibbons-Hawking term must not be added since it appears automatically as we integrate the Einstein-Hilbert term over the thin layers [36]. On the other hand, if I_5 does not include the integration over the thin layers then the Gibbons-Hawking boundary term must be added. In all cases, including the variation of the metric, the position of the hypersurface and the scalar field, the variational principle gives the correct set of equations of motion: the Einstein equation and the field equation of the scalar field off the branes, Israel's junction condition and the matching condition for the scalar field [36].

A. Bulk equation

For simplicity we assume that the geometry on each brane is everywhere and for all times described by the 4D flat FRW universe and that the two branes are parallel to each other. In this case, it is natural to expect that the 5D bulk spacetime between the two branes also has the same symmetry. Hence, we assume that the metric and the scalar field have the following forms:

$$ds^2=-n(t,y)^2dt^2+a(t,y)^2\delta_{ij}dx^idx^j+b(t,y)^2dy^2,$$

$$\phi=\phi(t,y), \tag{6}$$

where $i=1,2,3$; y represents the extra dimension; and the world-volume of each brane is expressed as

$$y=Y_\pm(t) \quad [Y_-(t)<Y_+(t)]. \tag{7}$$

For this ansatz, we obtain

$$\frac{G_{tt}}{n^2}=\frac{3}{n^2}\left[\left(\frac{\dot a}{a}\right)^2+\frac{\dot a}{a}\frac{\dot b}{b}\right]+\frac{3}{b^2}\left[-\frac{a''}{a}-\left(\frac{a'}{a}\right)^2+\frac{a'}{a}\frac{b'}{b}\right],$$

$$G_{ty}=3\left[-\frac{\dot a'}{a}+\frac{\dot a}{a}\frac{n'}{n}+\frac{a'}{a}\frac{\dot b}{b}\right],$$

$$\frac{G_{yy}}{b^2}=\frac{3}{n^2}\left[-\frac{\ddot a}{a}-\left(\frac{\dot a}{a}\right)^2+\frac{\dot a}{a}\frac{\dot n}{n}\right]+\frac{3}{b^2}\left[\left(\frac{a'}{a}\right)^2+\frac{a'}{a}\frac{n'}{n}\right],$$

$$\frac{G_{ij}}{a^2}=\frac{\delta_{ij}}{n^2}\left[-2\frac{\ddot a}{a}-\left(\frac{\dot a}{a}\right)^2+2\frac{\dot a}{a}\frac{\dot n}{n}-2\frac{\dot a}{a}\frac{\dot b}{b}+\frac{\dot b}{b}\frac{\dot n}{n}-\frac{\ddot b}{b}\right]$$
$$+\frac{\delta_{ij}}{b^2}\left[2\frac{a''}{a}+\left(\frac{a'}{a}\right)^2+2\frac{a'}{a}\frac{n'}{n}-2\frac{a'}{a}\frac{b'}{b}-\frac{b'}{b}\frac{n'}{n}+\frac{n''}{n}\right],$$

$$G_{ti}=G_{yi}=0, \tag{8}$$

and

$$\frac{T_{tt}}{n^2}=\frac{1}{2}\left(\frac{\dot\phi}{n}\right)^2+\frac{1}{2}\left(\frac{\phi'}{b}\right)^2+V,$$

$$T_{ty}=\dot\phi\phi',$$

$$\frac{T_{yy}}{b^2}=\frac{1}{2}\left(\frac{\dot\phi}{n}\right)^2+\frac{1}{2}\left(\frac{\phi'}{b}\right)^2-V,$$

$$\frac{T_{ij}}{a^2}=\frac{1}{2}\left(\frac{\dot\phi}{n}\right)^2-\frac{1}{2}\left(\frac{\phi'}{b}\right)^2-V,$$

$$T_{ti}=T_{yi}=0, \tag{9}$$

where a dot and a prime denote derivatives with respect to t and y, respectively.

B. Boundary condition

On each brane at $y=Y_\pm(t)$, the induced metric $q_{\pm\mu\nu}$, the extrinsic curvature $K_{\pm\mu\nu}$ and the pullback $\phi_\pm$ of the scalar field play important roles. They are given by

$$
\begin{aligned}
&q_{\pm tt}=-[n^2-b^2\dot{Y}_\pm^2],\\
&q_{\pm ij}=a^2\delta_{ij},\\
&\mathcal{K}_\pm\equiv K_{\pm\,t}^{\ t}=\left[1-\left(\frac{b\dot{Y}_\pm}{n}\right)^2\right]^{-3/2}\left[\frac{b\ddot{Y}_\pm}{n^2}-\left(\frac{b\dot{Y}_\pm}{n}\right)^3\frac{\dot{b}}{bn}\right.\\
&\qquad\left.+\left(\frac{b'}{b^2}-\frac{2n'}{bn}\right)\left(\frac{b\dot{Y}_\pm}{n}\right)^2+\left(\frac{2\dot{b}}{bn}-\frac{\dot{n}}{n^2}\right)\frac{b\dot{Y}_\pm}{n}+\frac{n'}{bn}\right],\\
&\bar{\mathcal{K}}_\pm\delta^i_j\equiv K_{\pm\,j}^{\ i}=\delta^i_j\left[1-\left(\frac{b\dot{Y}_\pm}{n}\right)^2\right]^{-1/2}\left[\frac{a'}{ab}+\frac{\dot{a}}{an}\frac{b\dot{Y}_\pm}{n}\right],\\
&\phi_\pm=\phi(t,Y_\pm),\\
&\partial_\perp\phi_\pm\equiv n^\mu\partial_\mu\phi|_{y=Y_\pm(t)}=\left[1-\left(\frac{b\dot{Y}_\pm}{n}\right)^2\right]^{-1/2}\left[\frac{b\dot{Y}_\pm}{n}\frac{\dot{\phi}}{n}\right.\\
&\qquad\left.+\frac{\phi'}{b}\right].
\end{aligned}
\tag{10}
$$

Having the Z_2 symmetry, the Israel junction condition and the scalar field matching condition are written as

$$
\begin{aligned}
\mathcal{K}_\pm&=\pm\frac{\kappa_5^2}{6}\lambda_\pm(\phi_\pm),\\
\bar{\mathcal{K}}_\pm&=\pm\frac{\kappa_5^2}{6}\lambda_\pm(\phi_\pm),\\
\partial_\perp\phi_\pm&=\mp\frac{1}{2}\partial_{\phi_\pm}\lambda_\pm(\phi_\pm).
\end{aligned}
\tag{11}
$$

When $Y_\pm(t)$ are constants $y_\pm$, these conditions reduce to the following simple conditions

$$\left.\frac{n'}{bn}\right|_{y=y_\pm}=\left.\frac{a'}{ba}\right|_{y=y_\pm}=\pm\frac{\kappa_5^2}{6}\lambda_\pm(\phi_\pm),\tag{12}$$

$$\left.\frac{\phi'}{b}\right|_{y=y_\pm}=\mp\frac{1}{2}\partial_{\phi_\pm}\lambda_\pm(\phi_\pm).\tag{13}$$

IV. SCALING SOLUTION

In this section we shall seek a scaling solution for a specific model with exponential potentials. For a special choice of parameters, the model reduces to the 5D reduction of the Horava-Witten theory [37]. On the other hand, the solution we shall find does not appear to reduce to the known flat FRW solution in the Horava-Witten theory [38]. (For closed and open FRW solutions in Horava-Witten theory, see Ref. [39].)

A. Action

Exponential potentials of the form

$$V=V_0e^{-2\alpha\kappa\phi}\tag{14}$$

for (dilaton coupling) constant α, arise in many theories of the fundamental interactions including superstring and higher-dimensional theories. Typically, "realistic" supergravity theories predict steep exponential potentials [40] (i.e., $2\alpha^2>1$). The effective action of the Horava-Witten theory has $\alpha^2=2$ (see below). A smaller *effective* α can arise in assisted theories which contain more than one scalar field [41]. In addition, for many compactifications of higher-dimensional theories there is a consistent truncation to a single scalar field ϕ and positive potentials of this form which arises via generation by nonzero flux of antisymmetric tensor fields (the T^7 compactification of 11D supergravity with nonvanishing 4-form field strength yields an exponential potential with $\alpha=\sqrt{7}$; in general, flux compactifications seem to yield $\alpha\geqslant\sqrt{3}$ [42]), and hyperbolic compactifications in the context of string/M theory [43,44] (the compactification of 11D supergravity on a 7D compact hyperbolic space has $\alpha=3/\sqrt{7}$; in general, hyperbolic compactifications seem to lead to $1<\alpha<\sqrt{3}$ [45]). A comprehensive qualitative analysis of scalar field cosmological models with an exponential potential has been presented [46,47].

In this paper we consider a brane-world analogue of such models with exponential potentials. To motivate our model with exponential potentials, let us briefly review the 5D reduction of the Horava-Witten theory. The 5D effective action for the Horava-Witten theory is [37]

$$
\begin{aligned}
I_{HW}=&\frac{1}{2\kappa_5^2}\int dx^5\sqrt{-g}\left[R-\frac{\partial^M\Phi\partial_M\Phi}{2\Phi^2}-\frac{\tilde{\alpha}^2}{3\Phi^2}\right]\\
&+\frac{\sqrt{2}}{\kappa_5^2}\int dx_-^4\sqrt{-q_-}\,\frac{\tilde{\alpha}}{\Phi_-}-\frac{\sqrt{2}}{\kappa_5^2}\int dx_+^4\sqrt{-q_+}\,\frac{\tilde{\alpha}}{\Phi_+},
\end{aligned}
\tag{15}
$$

where $\Phi_\pm$ is the pullback of the bulk scalar field Φ onto each brane. If we introduce a canonically normalized scalar field ϕ by

$$\sqrt{2}\kappa_5\phi\equiv\ln\left(\frac{\Phi}{\sqrt{2}\tilde{\alpha}\kappa_5^{2/3}}\right),\tag{16}$$

this action is reduces to

$$
\begin{aligned}
I_{HW}=&\int dx^5\sqrt{-g}\left[\frac{1}{2\kappa_5^2}R-\frac{1}{2}\partial^M\phi\partial_M\phi-\frac{e^{-2\sqrt{2}\kappa_5\phi}}{12\kappa_5^{10/3}}\right]\\
&+\int dx_-^4\sqrt{-q_-}\,\frac{e^{-\sqrt{2}\kappa_5\phi_-}}{\kappa_5^{8/3}}\\
&-\int dx_+^4\sqrt{-q_+}\,\frac{e^{-\sqrt{2}\kappa_5\phi_+}}{\kappa_5^{8/3}},
\end{aligned}
\tag{17}
$$

where $\phi_\pm$ is the pullback of the scalar field ϕ on each brane.

The model we shall consider is

$$I=\int dx^5\sqrt{-g}\left[\frac{1}{2\kappa_5^2}R-\frac{1}{2}\partial^M\phi\partial_M\phi-V_0e^{-2\alpha\kappa_5\phi}\right]-\int dx_-^4\sqrt{-q_-}\lambda_0^-e^{-\alpha\kappa_5\phi_-}-\int dx_+^4\sqrt{-q_+}\lambda_0^+e^{-\alpha\kappa_5\phi_+}, \tag{18}$$

where α, V_0 and $\lambda_0^\pm$ are constants. This action includes the 5D effective action of the Horava-Witten theory as a special case ($\alpha=\sqrt{2}$).

B. Ansatz

We would like to seek a special solution which corresponds to an "equilibrium state," which represents a future attractor for the model. In a general situation where branes move arbitrarily, the motion of branes produces waves of the scalar field ϕ in the bulk. The waves in the bulk interact with both branes in the sense that they can come and go between two end-of-the-world branes. Hence, the general situation is very complicated, and nonlocal from the viewpoint of a 4D observer on a brane [48]. On the other hand, we would like to seek a special solution in which waves emitted from one brane and those from another brane are in equilibrium so that there is effectively no scalar wave in the bulk. Hence, what we would like to seek is a very special situation in which the nonlocal effects due to scalar waves are completely suppressed.

In this kind of "equilibrium" situation without waves in the bulk, nothing should propagate from one brane to another and, thus, we expect that all physically meaningful functions of t and y should be a product of a function of t and a function of y. In particular, in this case the time dependence of n can be removed by a coordinate choice. Moreover, from our experience in 4D scalar field cosmology with an exponential potential, we expect that the "equilibrium" situation should correspond to a power-law expansion on each brane, $a(t,y_\pm)\propto t^p$ [46]. Actually, in 4D cosmology with an exponential potential, a power-law expansion is an attractor of the system. One of the essential physical reasons for this is that all physically relevant, dimensionful quantities scale in tandem, according to dimensionality. For example, $H^2\propto\kappa_4^2V\propto\kappa_4^2\rho_{kin}\propto 1/t^2$, where H is the Hubble expansion rate, κ_4 is the 4D gravitational coupling, V is the exponential potential and ρ_{kin} is the kinetic energy of the scalar field. In our model, besides the Hubble expansion rate on each brane, the interbrane distance is another physically relevant, dimensionful quantity. Since it has the dimension of length, we expect it should also be proportional to t in the "equilibrium" situation. Thus, not only the 4D universes on branes but also the interbrane distance should be expanding.

For the reasons explained above, we consider the following ansatz:

$$ds^2=n(y)^2(-dt^2+t^{2p}\delta_{ij}dx^idx^j+t^2dy^2),$$

$$\kappa_5^2|V_0|e^{-2\alpha\kappa_5\phi}=\frac{e^{-2\psi(y)}}{l^2t^2},$$

$$Y_\pm(t)=y_\pm, \tag{19}$$

where $n(y)$ and $\psi(y)$ are functions of y, and p, l and $y_\pm$ ($y_-<y_+$) are constants. Note that t, y and x^i are dimensionless and that $n(y)$ and l have the dimension of length.

C. Equations

With this ansatz, the Einstein equation in the bulk reduces to

$$3\alpha^2p-1=0,$$

$$(\ln n-p\psi)'=0,$$

$$(\psi')^2-1\pm\frac{6\alpha^4}{l^2(4-3\alpha^2)}n^2e^{-2\psi}=0, \tag{20}$$

where the plus and minus signs correspond to $V_0>0$ and $V_0<0$, respectively. Hence, we obtain

$$p=\frac{1}{3\alpha^2},$$

$$n=n_0e^{p\psi}, \tag{21}$$

where n_0 is an arbitrary constant and ψ is a solution to the following equation:

$$(\psi')^2-1\pm e^{2(p-1)\psi}=0, \tag{22}$$

where we have set

$$l=\sqrt{\frac{6}{|4-3\alpha^2|}}\alpha^2n_0, \tag{23}$$

and the plus and minus signs correspond to $(4-3\alpha^2)V_0>0$ and $(4-3\alpha^2)V_0<0$, respectively, and we have assumed that $\alpha^2\neq 4/3$.

The Israel junction condition (12) is written as

$$e^{-(p-1)\psi}\psi'|_{y=y_\pm}=\pm\gamma_\pm,$$

$$\gamma_\pm\equiv\frac{\alpha^2\kappa_5\lambda_0^\pm n_0}{2\sqrt{|V_0|}l}=\frac{\kappa_5\lambda_0^\pm}{2\sqrt{|V_0|}}\sqrt{\frac{|4-3\alpha^2|}{6}}. \tag{24}$$

The scalar field matching condition (13) is actually the same as the above junction condition and, thus, does not provide any independent boundary conditions.

The physically relevant quantities are the power index $p=1/3\alpha^2$ of the expansion of the branes, the warp factor $W\equiv n(y_+)/n(y_-)$, and the ratio $v_\pm$ of the interbrane distance

$L=t\int_{y_-}^{y_+}n(y)dy$ to the proper time $\tau_\pm=n(y_\pm)t$ on each brane at $y=y_\pm$. These are given by

$$W\equiv\frac{n(y_+)}{n(y_-)}=\frac{\tau_+}{\tau_-}=e^{p[\psi(y_+)-\psi(y_-)]},$$

$$v_\pm\equiv\frac{L}{\tau_\pm}=e^{-p\psi(y_\pm)}\int_{y_-}^{y_+}e^{p\psi(y)}dy. \tag{25}$$

D. Solutions

For $(4-3\alpha^2)V_0<0$, the solution to the bulk equation is

$$e^{-(p-1)\psi}=\begin{cases}\sinh[|p-1|(y-y_0)] & (\text{for } y_0<y)\\ \sinh[|p-1|(y_0-y)] & (\text{for } y<y_0),\end{cases} \tag{26}$$

where y_0 is a constant. Note that ψ' diverges at $y=y_0$. Hence, y_0 must not be between the two branes. For this solution, the junction condition reduces to

$$\cosh[(p-1)(y_\pm-y_0)]=\mp\gamma_\pm \quad (\text{for } y_0<y_-<y_+)$$
$$\cosh[(p-1)(y_\pm-y_0)]=\pm\gamma_\pm \quad (\text{for } y_-<y_+<y_0). \tag{27}$$

Hence, if and only if $1<\gamma_-<-\gamma_+$ or $1<\gamma_+<-\gamma_-$, the junction condition uniquely determines $y_\pm-y_0$. The physically relevant quantities W and $v_\pm$ are given by

$$W=\left(\frac{\gamma_-^2-1}{\gamma_+^2-1}\right)^{p/2(p-1)},$$

$$v_\pm=(\gamma_\pm^2-1)^{p/2(p-1)}\int_{y_-}^{y_+}\frac{dy}{[\sinh|(p-1)(y-y_0)|]^{p/(p-1)}}. \tag{28}$$

Not only W but also $v_\pm$ are uniquely determined by p (or equivalently $|\alpha|$) and $\gamma_\pm$. For example, when $p=2$,

$$\sqrt{v_+v_-}=|\gamma_+|\sqrt{\gamma_-^2-1}+|\gamma_-|\sqrt{\gamma_+^2-1}. \tag{29}$$

For $(4-3\alpha^2)V_0>0$, the solution to the bulk equation is

$$e^{-(p-1)\psi}=\cosh[(p-1)(y-y_0)], \tag{30}$$

where y_0 is a constant. For this solution, the junction condition reduces to

$$\sinh[(p-1)(y_\pm-y_0)]=\mp\gamma_\pm. \tag{31}$$

Hence, if and only if $\gamma_-+\gamma_+<0$ then the junction condition uniquely determines $y_\pm-y_0$. The physically relevant quantities W and $v_\pm$ are given by

$$W=\left(\frac{\gamma_-^2+1}{\gamma_+^2+1}\right)^{p/2(p-1)},$$

$$v_\pm=(\gamma_\pm^2+1)^{p/2(p-1)}\int_{y_-}^{y_+}\frac{dy}{[\cosh|(p-1)(y-y_0)|]^{p/(p-1)}}. \tag{32}$$

Again, not only W but also $v_\pm$ are uniquely determined by p (or equivalently $|\alpha|$) and $\gamma_\pm$. For example, when $p=2$,

$$\sqrt{v_+v_-}=-\gamma_+\sqrt{\gamma_-^2+1}-\gamma_-\sqrt{\gamma_+^2+1}. \tag{33}$$

V. STABILITY AGAINST LINEAR PERTURBATIONS

Now let us argue that the expanding scaling solution is a future attractor. For this purpose we investigate linear perturbations around the scaling solution and show stability. The linear perturbations analyzed in this section include all important physical effects such as boundary conditions on the branes, fluctuations of brane positions and scalar waves in the bulk. The bulk scalar waves generated by fluctuations of brane positions threaten to destabilize the expanding scaling solution, but we shall explicitly see that this is not the case and that the scaling solution is stable. In the following, for simplicity we consider the case where $\psi'(y)\neq 0$ for $y_-\leq y\leq y_+$.

Let us consider linear perturbations around the scaling solution found in the preceding section:

$$ds^2=-n(t,y)^2dt^2+a(t,y)^2\delta_{ij}dx^idx^j+b(t,y)^2dy^2+2\epsilon(h_{ty}dtdy+h_{ti}dtdx^i+h_{yi}dydx^i),$$

$$n(t,y)=n_0e^{p\psi(y)}[1+\epsilon\delta n(t,y)],$$

$$a(t,y)=t^pn_0e^{p\psi(y)}[1+\epsilon\delta a(t,y)],$$

$$b(t,y)=tn_0e^{p\psi(y)}[1+\epsilon\delta b(t,y)],$$

$$\alpha\kappa_5\phi(t,y)=\ln[\kappa_5\sqrt{|V_0|}\ \ lt]+\psi(y)+\epsilon\delta\psi(t,y), \tag{34}$$

where p and $\psi(y)$ are given in the preceding section. From the symmetry, we set

$$h_{ty}=h_{ti}=h_{yi}=0. \tag{35}$$

By introducing a new time coordinate

$$\tau\equiv\ln t \quad (-\infty<\tau<\infty), \tag{36}$$

it is shown that all coefficients in all relevant equations (i.e., the Einstein equation, the Israel junction condition and the scalar field matching condition) linearized with respect to ϵ are independent of τ. Thus, it is convenient to Fourier expand the perturbations as

$$\delta n(t,y)=\delta\overline{n}(y)e^{-i\omega\tau},$$

$$\delta a(t,y)=\delta\overline{a}(y)e^{-i\omega\tau},$$

$$\delta b(t,y)=\delta\overline{b}(y)e^{-i\omega\tau},$$

$$\delta\psi(t,y)=\delta\overline{\psi}(y)e^{-i\omega\tau}. \tag{37}$$

A. Gauge conditions

The set of variables ($h_{ty,ti,yi}$, δn, δa, δb, $\delta\psi$) includes not only physical degrees of freedom but also gauge degrees of freedom. The infinitesimal gauge transformation is

$$\delta g_{\mu\nu} \to \delta g_{\mu\nu} - \epsilon\nabla_\mu \xi_\nu - \epsilon\nabla_\nu \xi_\mu ,$$
$$\delta\phi \to \delta\phi - \epsilon\xi^\mu \nabla_\mu \phi^{(0)}, \tag{38}$$

where ∇ is the covariant derivative compatible with the background metric, $\phi^{(0)}$ is the background of ϕ and $\epsilon\xi_\mu$ is an infinitesimal vector representing the gauge degrees of freedom. Hence, each component transforms as

$$\overline{\delta n} \to \overline{\delta n} + \frac{1}{n_0^2}[(1-i\omega)\overline{\xi_t}(y) - p\psi'(y)\overline{\xi_y}(y)],$$
$$\overline{\delta a} \to \overline{\delta a} + \frac{1}{n_0^2}[p\overline{\xi_t}(y) - p\psi'(y)\overline{\xi_y}(y) - \overline{\xi_\parallel}(y)],$$
$$\overline{\delta b} \to \overline{\delta b} + \frac{1}{n_0^2}[\overline{\xi_t}(y) - p\psi'(y)\overline{\xi_y}(y) - \overline{\xi_y'}(y)],$$
$$h_{ty} \to h_{ty} + te^{2p\psi(y)}[i\omega\overline{\xi_y}(y) - \overline{\xi_t'}(y)],$$
$$h_{ti} \to h_{ti} + i\omega t^{2p-1} e^{2p\psi(y)}\overline{\xi_\parallel}(y)\delta_{ij}(x^j - x_0^j),$$
$$h_{yi} \to h_{yi} - t^{2p} e^{2p\psi(y)}\overline{\xi_\parallel'}(y)\delta_{ij}(x^j - x_0^j),$$
$$\overline{\delta\psi} \to \overline{\delta\psi} + \frac{1}{n_0^2}[\overline{\xi_t}(y) - \psi'(y)\overline{\xi_y}(y)], \tag{39}$$

where we have Fourier expanded ξ_μ as

$$\xi_t = te^{2p\psi(y)}\overline{\xi_t}(y)e^{-i\omega\tau},$$
$$\xi_y = t^2 e^{2p\psi(y)}\overline{\xi_y}(y)e^{-i\omega\tau}, \tag{40}$$
$$\xi_i = t^{2p} e^{2p\psi(y)}\overline{\xi_\parallel}(y)e^{-i\omega\tau}\delta_{ij}(x^j - x_0^j).$$

Here, x_0^j ($j=1,2,3$) are constants. The symmetry assumption $h_{ty}=h_{ti}=h_{yi}=0$ is consistent with the gauge transformation if and only if

$$i\omega\overline{\xi_y}(y) - \overline{\xi_t'}(y) = \omega\overline{\xi_\parallel}(y) = \overline{\xi_\parallel'}(y) = 0. \tag{41}$$

By using the gauge freedom, we can set

$$\overline{\delta\psi} = 0. \tag{42}$$

However, this condition does not fix the gauge freedom completely. So, let us investigate the residual gauge freedom.

(i) For $\omega \neq 0$ and $-i\omega \neq p-1$, the residual gauge freedom is

$$\overline{\delta n} \to \overline{\delta n} - \frac{i\omega + p - 1}{n_0^2}\overline{\xi}(y),$$
$$\overline{\delta a} \to \overline{\delta a},$$
$$\overline{\delta b} \to \overline{\delta b} - \frac{i\omega + p - 1}{[n_0\psi']^2}\overline{\xi}(y), \tag{43}$$

where $\overline{\xi}$ $(=\overline{\xi_t})$ is an arbitrary solution of

$$\psi'\overline{\xi'} = i\omega\overline{\xi}. \tag{44}$$

Here, we have used the background equation (22) and its derivative.

(ii) For $\omega \neq 0$ and $-i\omega = p-1$, there is no residual gauge freedom.

(iii) For $\omega = 0$ and $p \neq 1$, the residual gauge freedom is

$$\overline{\delta n} \to \overline{\delta n} + \overline{\delta n}_0 ,$$
$$\overline{\delta a} \to \overline{\delta a} + \overline{\delta a}_0 ,$$
$$\overline{\delta b} \to \overline{\delta b} + \frac{\overline{\delta n}_0}{[\psi'(y)]^2} \tag{45}$$

where δn_0 and δa_0 are constants.

(iv) For $\omega = 0$ and $p = 1$, the residual gauge freedom is

$$\overline{\delta a} \to \overline{\delta a} + \overline{\delta a}_0 , \tag{46}$$

where δa_0 is a constant.

B. Linearized equations

By using τ as a time coordinate, the Einstein equation linearized with respect to ϵ becomes relatively simple in the sense that there is no explicit τ dependence in any coefficients. The Israel junction condition and the scalar field matching condition also share this property. This is the reason why we have Fourier-expanded the linear quantities (δn, δa, δb, $\delta\psi$) with respect to τ. Hence, what we have is just an eigenvalue problem in one dimension.

The (1D) bulk equations are

$$\psi'\overline{\delta n'} = -\frac{\omega^2 + (p-1)i\omega}{p}\overline{\delta a} + (p-1)(\psi')^2\overline{\delta b} + [i\omega - (p-1)]\overline{\delta n},$$
$$\psi'\overline{\delta a'} = -i\omega\overline{\delta a} + p(\psi')^2\overline{\delta b} - p\overline{\delta n},$$
$$\psi'\overline{\delta b'} = -\frac{\omega^2 + (p-1)i\omega}{p}\overline{\delta a} + [i\omega + 2(p-1) - (4p-1)(\psi')^2]\overline{\delta b} + (2p+1)\overline{\delta n}, \tag{47}$$

where the plus and minus signs correspond to $(4p-1)V_0 > 0$ and $(4p-1)V_0 < 0$, respectively.

Since the gauge freedom has been fixed only up to the residual gauge freedom, the brane position is not at $y=y_\pm$ any more. Hence, we need to consider the perturbed position of the brane:

$$y=Y_\pm(\tau)\equiv y_\pm+\epsilon\overline{\delta y_\pm}e^{-i\omega\tau}. \tag{48}$$

For the fluctuating position of the brane, the formulas (12) and (13) cannot be used. As explained in the Appendix, the general boundary condition (11) gives the following boundary condition for perturbations:

$$\begin{aligned}\overline{\delta n'}&=i\omega\psi'\,\overline{\delta b},\\ \overline{\delta a'}&=0,\\ (p-1+i\omega)\overline{\delta y_\pm}&=-\psi'\,\overline{\delta b}.\end{aligned} \tag{49}$$

Note that these boundary conditions are imposed on $y=y_\pm$ [not $Y_\pm(y)$].

1. Second-order equation for $\overline{\delta a}$

From the set of equations (47) and the background equation (22), we can show that $\overline{\delta a}$ satisfies the following second-order ordinary differential equation:

$$\overline{\delta a''}+3p\psi'\,\overline{\delta a'}+\omega(\omega+3ip)\overline{\delta a}=0. \tag{50}$$

Now let us show that $\Im\omega<0$ unless $\overline{\delta a}\equiv 0$. By using Eq. (50) with the boundary condition $\overline{\delta a'}=0$ at $y=y_\pm$, it is easy to show that

$$\omega(\omega+3ip)\int_{y_-}^{y_+}dye^{3p\psi}|\overline{\delta a}|^2=\int_{y_-}^{y_+}dye^{3p\psi}|\overline{\delta a'}|^2. \tag{51}$$

Hence, $\omega(\omega+3ip)$ is real and non-negative unless $\overline{\delta a}\equiv 0$. The eigenvalue $\omega(\omega+3ip)$ can vanish only if $\overline{\delta a'}=0$. Thus,

$$-3p\leqslant\Im\omega<0, \tag{52}$$

or

$$\omega=0,\quad \overline{\delta a'}=0. \tag{53}$$

In the latter case, $\overline{\delta a}$ can be set to zero by using the residual gauge freedom (45) or (46). Therefore, we have shown that $-3p\leqslant\Im\omega<0$ or $\overline{\delta a}\equiv 0$.

2. Solution with $\overline{\delta a}=0$

When $\overline{\delta a}=0$, the set of first-order equations (47) reduces to

$$\begin{aligned}\psi'\,\overline{\delta n'}&=i\omega\overline{\delta n},\\ (\psi')^2\overline{\delta b}&=\overline{\delta n}.\end{aligned} \tag{54}$$

For $-i\omega\neq p-1$, the boundary condition (49) is automatically satisfied, but any solutions to (54) with $\overline{\delta a}=0$ is pure gauge because of the residual gauge freedom (43) or (45). On the other hand, for $-i\omega=p-1$, there is no residual gauge freedom, but the third boundary condition in (49) becomes $\overline{\delta b}=0$ at $y=y_\pm$ and, thus, (54) implies that $\overline{\delta b}=\overline{\delta n}=0$. Therefore, we have shown that there is no nonvanishing physical solution with $\overline{\delta a}=0$.

3. Stability

In summary, we have found that there is no unstable mode with $\overline{\delta a}\neq 0$ and that there is no physical mode with $\overline{\delta a}=0$. Thus, the scaling solution is stable against linear perturbations.

VI. TOWARDS A NONLINEAR STABILITY ANALYSIS

Following the linear perturbation stability analysis, we need to establish the stability at the nonlinear level. In general this is very complicated. However, we can investigate some of the nonlinear effects by considering a subclass of models with a particular ansatz. This will give us an indication of the qualitative features of the nonlinear dynamics. We shall perform a more comprehensive analysis at a later time.

The ansatz we consider is

$$\begin{aligned}ds^2&=n_0^2e^{2p\psi(y)}[-N(t)^2dt^2\\&\quad+A(t)^2\delta_{ij}dx^idx^j+B(t)^2dy^2],\\ \tilde{\alpha}\kappa_5\phi&=\Theta(t)+\psi(y),\end{aligned} \tag{55}$$

where n_0 is a constant with the dimension of length. The scaling solution is, of course, included within this ansatz. The (ty) component of Einstein equation leads to

$$\Theta(t)=\ln[B(t)]+\Theta_0, \tag{56}$$

where Θ_0 is an arbitrary constant. Since Θ_0 can be absorbed by a redefinition of $\psi(y)$, for convenience we can choose Θ_0 as

$$e^{2\Theta_0}=\frac{2\kappa_5^2n_0^2|V_0|}{3p|4p-1|}, \tag{57}$$

whence the field equation reduces to

$$
\begin{aligned}
&(1+2p)\frac{\ddot{A}}{A}-2(p-1)\frac{\dot{A}^2}{A^2}\\
&\qquad-\left[(4p-1)\frac{\dot{B}}{B}+(2p+1)\frac{\dot{N}}{N}\right]\frac{\dot{A}}{A}+3p^2\frac{\dot{B}^2}{B^2}=0,\\
&(1+2p)\frac{\ddot{B}}{B}+3p\frac{\dot{B}^2}{B^2}\\
&\qquad+\left[3(2p-1)\frac{\dot{A}}{A}-(2p+1)\frac{\dot{N}}{N}\right]\frac{\dot{B}}{B}-6\frac{\dot{A}^2}{A^2}=0,\\
&(\psi')^2\pm e^{2(p-1)\psi}\\
&\qquad-\frac{1}{p(2p+1)N^2}\left[2\frac{\dot{A}^2B^2}{A^2}+2\frac{\dot{A}B\dot{B}}{A}-p\dot{B}^2\right]=0,
\end{aligned}
\tag{58}
$$

where the plus and minus signs in the last equation are for $(4p-1)V_0>0$ and $(4p-1)V_0<0$, respectively. Hence, the model is *separable*.

Hereafter, we set $N(t)$ to a constant by choice of time coordinate. Defining

$$a=\frac{\dot{A}}{A},\quad b=\frac{\dot{B}}{B},\tag{59}$$

the last equation in (58) becomes

$$c_0^2B^{-2}=2a^2+2ab-pb^2,\tag{60}$$

where c_0^2 is effectively the rescaled separation constant, and the evolution equations are then

$$(1+2p)\dot{a}=-3a^2+(4p-1)ab-3p^2b^2,\tag{61}$$

$$(1+2p)\dot{b}=6a^2-3(2p-1)ab-(1+5p)b^2.\tag{62}$$

Differentiating Eq. (60) and using Eqs. (61) and (62) we get an expression which is satisfied identically, so that (60) is a constraint (that propagates along the solution curves).

We first note that since the system (61), (62) is homogeneous, we can define

$$x=a/b,\quad \overline{b}=\ln b,\tag{63}$$

and the system reduces to a single ordinary differential equation:

$$\frac{dx}{d\overline{b}}=\frac{6(x-p)\left[x+\frac{1}{2}(1+\sqrt{1+2p})\right]\left[x+\frac{1}{2}(1-\sqrt{1+2p})\right]}{(1+5p)+3(2p-1)x-6x^2},\tag{64}$$

which can be integrated to obtain

$$
\begin{aligned}
b=b_0^2(x-p)^{\tilde{\alpha}}&\left[x+\frac{1}{2}(1+\sqrt{1+2p})\right]^{\tilde{\beta}}\\
&\times\left[x+\frac{1}{2}(1-\sqrt{1+2p})\right]^{\tilde{\gamma}},
\end{aligned}
\tag{65}
$$

where b_0^2 is an integration constant and

$$
\begin{aligned}
\tilde{\alpha}&\equiv\frac{1}{3p},\\
\tilde{\beta}&\equiv\frac{-3p-1+\sqrt{2p+1}}{6p}\\
&=-\frac{(3\sqrt{2p+1}+1)(\sqrt{2p+1}-1)}{12p}\\
&=-1+\frac{(3\sqrt{2p+1}+5)(\sqrt{2p+1}-1)}{12p},\\
\tilde{\gamma}&\equiv\frac{-3p-1-\sqrt{2p+1}}{6p}\\
&=-\frac{(3\sqrt{2p+1}-1)(\sqrt{2p+1}+1)}{12p}\\
&=-1+\frac{(3\sqrt{2p+1}-5)(\sqrt{2p+1}+1)}{12p}.
\end{aligned}
\tag{66}
$$

Note that

$$
\begin{aligned}
&\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma}=-1,\\
&\tilde{\alpha}>0,\quad -1<\tilde{\beta}<0,\quad \tilde{\gamma}<0
\end{aligned}
\tag{67}
$$

for any positive $p=1/(3\alpha^2)$, and $\tilde{\gamma}>-1$ if $p>8/9$.

We can analyze the asymptotic form of (65), but perhaps a better way to show the late-time stability of the scaling solution is as follows. We rewrite Eq. (60) as

$$1-\frac{(2p+1)b^2}{4\left(a+\frac{1}{2}b\right)^2}=\frac{c_0^2B^{-2}}{2\left(a+\frac{1}{2}b\right)^2}.\tag{68}$$

Defining

$$z=\frac{b\sqrt{1+2p}}{2a+b}=\frac{\sqrt{1+2p}}{1+2x},\tag{69}$$

we see that z is bounded; $1-z^2\geqslant 0$. Using (65), the evolution equations (61) and (62) then become a single ordinary differential equation for z:

$$\dot{z}=\frac{3b_0^2}{2^{2\tilde{\beta}}\sqrt{1+2p}}(1-\sqrt{1+2p}z)^{\tilde{\alpha}+1}(1+z)^{\tilde{\beta}+1}(1-z)^{\tilde{\gamma}+1}.\tag{70}$$

Since $z^2 \leqslant 1$, this constitutes a one-dimensional nonlinear dynamical system.

The exponent $(\tilde{\alpha}+1)$ is positive definite, so that

$$z_s = \frac{1}{\sqrt{1+2p}} \tag{71}$$

(where $0 < z_s < 1$), is always an equilibrium point of (70). Depending on the signs of the exponents $(\tilde{\beta}+1)$ and $(\tilde{\gamma}+1)$, $z_\pm = \pm 1$ are also equilibrium points (z_- is always an equilibrium point, while z_+ is an equilibrium point for $p > 8/9$).

The solution of (70) close to z_s is given by

$$z = z_s - z_0^2 (t+t_0)^{-1/\tilde{\alpha}}, \tag{72}$$

so that $z \to z_s$ as $t \to \infty$ since $\tilde{\alpha} > 0$. Hence, z_s is a local sink. Since z is bounded, z_s is a global attractor in the physical phase space. Indeed, since $\tilde{\beta} < 0$ and $\tilde{\gamma} < 0$, $z_\pm$ act as local sources and all physical solutions evolve from one of z_+ and z_- to z_s.

From Eqs. (69) and (71), the global attractor has

$$\frac{a}{b} (=x) = p \tag{73}$$

so that Eq. (60) yields

$$\dot{B}^2 = \frac{c_0^2}{p(2p+1)} = \mathrm{const} \tag{74}$$

and hence after a time translation

$$B = B_0 t, \quad A = A_0 t^p, \tag{75}$$

so that the global attractor is the scaling solution.

To connect with earlier work, the approximate solution of (61) and (62) at late times [i.e., the linearized solution around the attractor (75)] is ($N \equiv 1$)

$$A = A_0 t^p [1 + 3pc_1 t^{-1} + c_2 t^{-3p}],$$

$$B = B_0 t \left[1 - \frac{c_1}{p} t^{-1} - \frac{c_2}{p} t^{-3p} \right], \tag{76}$$

where the c_i are arbitrary constants [and the constants are subject to the constraint (60); e.g., $c_0^2 = p(1+2p)B_0^2$]. From this we immediately see the decaying modes and the local stability of the attractor.

Consequently, we have an approximate bulk (5D) solution at late times. The 5D Ricci tensor is given by its attractor values of order $O(t^{-1})$ and $O(t^{-3p})$ (leading to $\rho_{kin} \sim t^{-2}$), and the 5D conformal tensor (which is zero for the exact attractor solution) can be evaluated to leading order. We can therefore calculate the irreducible decompositions $\mathcal{U}$, $\mathcal{Q}_\mu$, and $\mathcal{P}_{\mu\nu}$ of the projected bulk Weyl tensor $\mathcal{E}_{\mu\nu}$ [9,10], from which we find that

$$\mathcal{Q}_\mu = 0, \quad \mathcal{P}_{\mu\nu} = 0 \tag{77}$$

and

$$\mathcal{U} \sim [O(t^{-3p}) + O(t^{-2})] \rho_{kin} \leqslant \rho_{kin}. \tag{78}$$

VII. DISCUSSION

We have investigated a class of scale-invariant effective theories of 5D brane-world cosmology with a bulk scalar between two end-of-the-world branes. As a special case, this class includes the 5D reduction of Horava-Witten theory. We have found an exact, analytic scaling solution in which the scale factor of our 4D universe and the interbrane distance expand as t^p ($p > 0$) and t, respectively (see *Note added*). It is perhaps worth mentioning that the scaling solution with $p > 1$ corresponds to a power-law inflation on our brane. The scaling solution corresponds to an equilibrium state in which the motion of the branes does not produce any scalar waves in the bulk. Because of this remarkable physical property, the scaling solution is expected to be a future attractor of the system. Indeed, we have presented convincing evidence for the attractor behavior: the stability of the scaling solution against general linear perturbations with the 4D FRW symmetry and stability within a class of nonlinear perturbations.

Based on the attractor behavior of the scaling solution, in Sec. II we proposed a self-consistent and dynamical scenario in the early brane-world universe. First, since the scaling solution is a future attractor, the system is automatically driven towards it as far as the effective action at high energy is of the scale invariant form. Second, as the energy scale becomes sufficiently low according to the evolution along the scaling solution, the scale invariance of the effective action should be broken at some point. After that, the brane-world system deviates from the scaling behavior and starts to be driven by the radion stabilization. The radion stabilization guarantees that 4D gravity on our brane is described by Einstein theory and the standard cosmology is realized at low energy.

In this scenario, the scaling solution plays a central role: the attractor behavior of the scaling solution makes it possible to give well-defined initial conditions both in the bulk and on the brane for the evolution after the breaking of scale invariance. It is also the scaling solution that brings the system from high energy to low energy both in the bulk and on the brane and, thus, triggers the breaking of the scaling invariance.

Having a well-defined initial condition given by the scaling solution, a natural question arises: "What kind of evolution can we expect subsequently?" One interesting possibility is an epoch of inflation driven by the radion (i.e., the interbrane distance). As stated above, after the scale invariance of the effective theory is broken at some point, the radion stabilization mechanism takes over and starts driving the brane-world system. Indeed, we have suggested a simple illustrative action in which the transition from the scale invariance to the radion stabilization is smooth. During the transition, the dynamics of the bulk scalar field and the radion can drive the evolution of our 4D Universe. Of course, even without the scaling behavior at high energy, the bulk scalar and the radion can drive the system. A big difference, however, is that one cannot expect a well-defined, predictable initial condition in the case without the high-energy scaling behavior since the evolution from an arbitrary state to the radion stabilization should be a very violent process and should involve fully nonlinear (scalar and gravitational) waves in the bulk. Our scenario based on the high-energy scaling behavior makes the evolution towards the radion stabilization much smoother and, more importantly, predictable. In particular, in our scenario, there may be an inflation in the transition epoch. This possibility is certainly an interesting subject for the future work.

Another question that might naturally be asked is: ''What is the beginning of the brane world before the scaling behavior?'' Actually, since the attractor behavior of the scaling solution would make the low-energy evolution of the brane cosmology almost insensitive to the beginning, this question might be irrelevant for our scenario. Nonetheless, it may be still interesting to think about this kind of question. As already explained, for the scaling solution both the scale factor of our universe and the interbrane distance expand. This implies that both our 4D Universe and the extra dimension were extremely small at an early epoch. This is a completely trivial statement in our scenario, but it would not be so trivial if we did not have the high-energy scaling behavior. Without the high-energy scaling behavior, the initial value of the inter-brane distance for the evolution towards the stable value can be either small or large. On the other hand, in our scenario the initial value of the interbrane distance should start from a small value. Is this favorable from the viewpoint of quantum gravity? Can we consider the smallness of the initial inter-brane distance as an indication of a brane collision or brane scattering?[3] The last two questions are evidently outside the scope of this paper but will be interesting future projects.

It is perhaps worthwhile asking whether we can expect similar scenarios to work in more general situations and/or theories. When the dynamical effects of fields are negligible (e.g., the bulk waves), then we might expect the same qualitative behavior and hence the scenario to work. For example, we expect predictable, smooth evolution towards the radion stabilization to be valid in more general situations whenever a symmetry at high energy exists, leading to attractor behavior, which is subsequently broken by radion stabilization at low energy.

As already stressed many times, we have presented convincing evidence that the scaling solution is a future attractor. Since the attractor behavior is so essential in our scenario, in future work we shall further investigate the full nonlinear dynamics both analytically and numerically. We shall also include other matter fields on the brane. Since the scale invariant theory is motivated by compactification of higher-dimensional theories (see Secs. II and IV A), it is also worth investigating higher-dimensional interpretations of the scaling solution. In particular, it is interesting to generalize the scaling behavior to higher dimension (greater than 5) and study the resulting physical consequences.

Note added. Recently, the authors were informed of Ref. [49], in which a closely related solution had been found.

[3]It is perhaps interesting to note that the expanding scaling solution we found is a conformally Kasner geometry and that Kasner-like geometries were found to be generic collapsing solutions in brane collisions [33].

ACKNOWLEDGMENTS

The authors would like to thank Paolo Creminelli, Andrei Frolov, Lev Kofman, and Toby Wiseman for their useful comments. S.M. would like to acknowledge the kind hospitality of Dalhousie University where this work was initiated during his visit. Part of this work was done during S.M.'s visits to University of Toronto, University of Victoria and University of Alberta. He would be grateful to Lev Kofman, Werner Israel, and Valeri Frolov for their warm hospitality and stimulating discussions during the visits. S.M.'s research was supported in part by NSF grant PHY-0201124. A.A.C.'s research was funded by the Natural Sciences and Engineering Research Council of Canada.

APPENDIX: BOUNDARY CONDITION FOR LINEAR PERTURBATIONS

In this appendix we explain the derivation of the boundary condition (49) for linear perturbations. For a more general prescription, see Ref. [50]. For general trajectories of branes the boundary condition is given by (11) at $y = Y_\pm(t)$. For

$$\begin{aligned} n(t,y) &= n_0 e^{p\psi(y)}[1+\epsilon\delta\overline{n}(y)e^{-i\omega\tau}],\\ a(t,y) &= t^p n_0 e^{p\psi(y)}[1+\epsilon\delta\overline{a}(y)e^{-i\omega\tau}],\\ b(t,y) &= t n_0 e^{p\psi(y)}[1+\epsilon\delta\overline{b}(y)e^{-i\omega\tau}],\\ \alpha\kappa_5\phi(t,y) &= \ln[\kappa_5\sqrt{|V_0|}\;\; lt]+\psi(y),\\ Y_\pm(\tau) &= y_\pm+\epsilon\delta\overline{y}_\pm e^{-i\omega\tau}, \end{aligned} \tag{A1}$$

the extrinsic curvature components $\mathcal{K}$ and $\overline{\mathcal{K}}$, and the normal derivative $\partial_\perp\phi_\pm$ of the scalar field are expanded as

$$\begin{aligned} \mathcal{K}_\pm &= \mathcal{K}_\pm^{(0)}+\epsilon\mathcal{K}_\pm^{(1)}+O(\epsilon^2),\\ \overline{\mathcal{K}}_\pm &= \overline{\mathcal{K}}_\pm^{(0)}+\epsilon\overline{\mathcal{K}}_\pm^{(1)}e^{-i\omega\tau}+O(\epsilon^2),\\ \partial_\perp\phi_\pm &= \partial_\perp\phi_\pm^{(0)}+\epsilon\partial_\perp\phi_\pm^{(1)}e^{-i\omega\tau}+O(\epsilon^2), \end{aligned} \tag{A2}$$

where

$$\begin{aligned} \mathcal{K}_\pm^{(0)} = \overline{\mathcal{K}}_\pm^{(0)} &= \frac{p\psi' e^{-p\psi}}{n_0 t},\\ \partial_\perp\phi_\pm^{(0)} &= \frac{\psi' e^{-p\psi}}{\alpha\kappa_5 n_0 t}, \end{aligned} \tag{A3}$$

and

$$\mathcal{K}_{\pm}^{(1)}=\frac{e^{-p\psi}}{n_0 t}[-(i\omega+\omega^2)\delta\overline{y}_{\pm}+\delta\overline{n'}-p\psi'\delta\overline{b}],$$

$$\overline{\mathcal{K}}_{\pm}^{(1)}=\frac{e^{-p\psi}}{n_0 t}[-ip\omega\delta\overline{y}_{\pm}+\delta\overline{a'}-p\psi'\delta\overline{b}],$$

$$\partial_{\perp}\phi_{\pm}^{(1)}=\frac{e^{-p\psi}}{\alpha\kappa_5 n_0 t}[-i\omega\delta\overline{y}_{\pm}-\psi'\delta\overline{b}]. \tag{A4}$$

What is important here is that the boundary condition in the form (11) must be imposed on the perturbed position $y=Y_{\pm}(t)$ of the brane. On the other hand, it is mathematically convenient to impose a boundary condition on a fixed position in the coordinate y. Hence, let us convert the boundary condition (11) at $y=Y_{\pm}(t)$ to a boundary condition at $y=y_{\pm}$. The result up to the linear order in ϵ is

$$\mathcal{K}_{\pm}^{(0)\prime}\delta\overline{y}_{\pm}+\mathcal{K}_{\pm}^{(1)}=\pm\frac{\kappa_5}{6\alpha}(\partial_{\phi_{\pm}}\lambda_{\pm})\psi'\delta\overline{y}_{\pm},$$

$$\overline{\mathcal{K}}_{\pm}^{(0)\prime}\delta\overline{y}_{\pm}+\overline{\mathcal{K}}_{\pm}^{(1)}=\pm\frac{\kappa_5}{6\alpha}(\partial_{\phi_{\pm}}\lambda_{\pm})\psi'\delta\overline{y}_{\pm},$$

$$(\partial_{\perp}\phi_{\pm}^{(0)})'\delta\overline{y}_{\pm}+\partial_{\perp}\phi_{\pm}^{(1)}=\mp\frac{1}{2\alpha\kappa_5}(\partial_{\phi_{\pm}}^2\lambda_{\pm})\psi'\delta\overline{y}_{\pm}. \tag{A5}$$

This form of the boundary condition must be imposed at $y=y_{\pm}$. Substituting the above expressions for $K_{\pm}^{(i)}$, $\overline{K}_{\pm}^{(i)}$ and $\partial_{\perp}\phi_{\pm}^{(i)}$ and simplifying the expressions by using the background equations, we obtain the boundary condition (49).

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