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Viscous fluid collapse

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The problem of seeking solutions of Einstein's field equations that represent the collapse of realistic matter distributions is discussed. A specialized approach to this problem is taken in which the fact that a given energy-momentum tensor may formally represent different types of matter distribution is exploited. A solution is presented in which an "interior" solution consisting of a collapsing viscous fluid (i.e., a solution of the Einstein field equations for an imperfect fluid source) is matched continuously across its boundary to a Schwarzschild "exterior." In this solution the geometrical part corresponding to the interior solution is formally identical to that of a closed (i.e., $k=+1$) Friedmann-Robertson-Walker dust model.

INTRODUCTION

An important problem in current gravity theory is the search for solutions of Einstein's field equations which represent the collapse of realistic astronomical matter distributions to condensed objects, such as black holes. This problem takes the following form: A solution of the field equations with a nonzero energy-momentum tensor (interior solution) representing the interior of an astronomical body collapsing in upon itself and a solution of the vacuum field equations (exterior solution) representing the region external to the body are taken and matched continuously across the (collapsing) boundary of the object.

To date, this search has been rather fruitless in that only very simple interior solutions representing unrealistic matter distributions have been investigated. The standard model of collapse is given by the following (see Weinberg[1]). The interior is assumed to be represented by a closed (i.e., $k=+1$) Friedmann-Robertson-Walker (FRW) dust model. That is, the matter is represented by a pressure-free perfect fluid which is both isotropic and homogeneous. The Einstein field equations

$$G_{\mu\gamma}=\bar{\rho}(t)u_\mu u_\gamma \tag{1}$$

then have the solution represented by the FRW line element

$$ds^2=-dt^2+R^2(t)\left[\frac{dr^2}{1-r^2}+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2\right] \tag{2}$$

in a comoving coordinate system in which $0\leq r\leq 1$ and $0\leq t\leq \pi R_0$ (or at least $\pi R_0/2\leq t\leq \pi R_0$ which represents the "collapsing stage"). The actual solution is given parametrically by

$$t=\tfrac{1}{2}C(\psi-\sin\psi),\quad R=\tfrac{1}{2}C(1-\cos\psi)\ , \tag{3}$$

where we note that

$$\dot{R}^2=\frac{C}{R}-1,\quad \bar{\rho}=3CR^{-3},\quad \bar{p}=0\ , \tag{4}$$

and $C=R_0$ is the maximum value of R.

The collapse occurs at $\psi=2\pi$ or $t=\pi R_0$ when R becomes equal to zero. The collapsing model is completed by choosing $M/4\pi r_S{}^3=1$ where $r=r_S$ is the radius of the astronomical body (a constant, since r is a comoving coordinate), whence the solution matches to the Schwarzschild exterior vacuum solution, in that the gravitational field is continuous across the surface $r=r_S$.

The standard model above is inadequate because the astronomical body cannot be realistically represented by dust. The question arises as to whether models can be found in which the interior can be modeled by a more physical fluid, that of a viscous magnetohydrodynamic fluid with energy-momentum tensor given by

$$M_{\mu\gamma}=E_{\mu\gamma}+(\rho+p)u_\mu u_\gamma+pg_{\mu\gamma}$$
$$-2\eta\sigma_{\mu\gamma}+q_\mu u_\gamma+q_\gamma u_\mu\ , \tag{5}$$

where ρ, p, u^μ, $\sigma_{\mu\gamma}$, q_μ, and η are, respectively, the density, thermodynamic pressure, fluid velocity vector, shear tensor, heat-conduction vector, and shear-viscosity coefficient. $E_{\mu\gamma}$ is the energy-momentum tensor of any electromagnetic field present and is given by

$$E_{\mu\gamma}=F_{\mu\alpha}F_\gamma{}^\alpha-\tfrac{1}{4}g_{\mu\gamma}F_{\alpha\beta}F^{\alpha\beta}\ ,$$

where $F_{\mu\gamma}$ is the electromagnetic field strength tensor.

This problem is, however, very difficult. First, there are very few known viscous-fluid solutions of Einstein's field equations. Second, if we have a nondust interior solution, we cannot necessarily match this to an exterior Schwarzschild solution; we wish the nonperfect fluid quantities such as the electromagnetic field and the heat conduction to be continuous across the boundary of the astronomical body (in addition to the thermodynamical quantities). Therefore, we may need to use more "realistic" exterior solutions, such as Reissner-Nordström-type exteriors or radiation models [such as the $R(t)=t^{1/2}$, $k=0$ FRW model]. The more complicated interior and exterior solutions will in turn lead to more complicated junction conditions at the boundary of the astronomical body ("matchup" conditions).

A further difficulty that may arise with more complicated interior and exterior solutions concerns the types of coordinate systems employed in these solutions. In the standard model comoving coordinates are employed and the matchup occurs at $r=\text{constant}$. In more complicated solutions there may be difficulties in finding a suitable coordinate system for both interior and exterior solutions. In addition, if the coordinate systems are not comoving, difficulties arise concerning the speed of matter inside, on, and outside the boundary, and how to actually identify the boundary where the matchup occurs.

Another consideration when attempting more realistic solutions concerns the nature of the matchup itself. Perhaps there should be several regions, each characterizing a particular aspect of the nature of the star (or other astronomical body under investigation), and each of which could be matched up at their common boundary. One might imagine a central, dense core region, in which nuclear reactions are taking place and the model of which would necessarily contain quantum effects. This might match up to a region that could loosely be called the outer layers of the star. This might consist of a single region or several subregions. For example, we might have a viscous magnetohydrodynamic fluid deep inside the star matching up to a dustlike region representing the outermost layers of the star. This region then matches up at the boundary of the star to an exterior region (or perhaps, if the outer layers are dust, we might model the star as being a nonlocalized object; that is, the outer layer of the star does not have a sharp boundary, but it is dust whose density simply decreases as one moves away from the interior of the star).

These last considerations can actually be used to one's advantage, however. An important problem then becomes matching regions of the star to other regions so that we need not seek solutions which are true for all coordinate values, but only for coordinate values representative of the region under consideration. For example, we might seek a solution matching a viscous-fluid outer layer to an exterior solution, and examine the collapse of the outer layers of the star. In this scenario the viscous-fluid solution need not be valid down to $r=0$, but only down to $r=r_c$, where the model would then be assumed to match onto a central core region.

APPROACH/THE MODEL

We shall present one particular approach to the above problem. We will obtain a solution that will overcome all the above difficulties, although it will be a very specialized solution.

The approach is to exploit the fact that a given energy-momentum tensor may formally represent different types of matter distribution (Tupper[2,3] and Coley and Tupper[4–6]). That is, the energy-momentum tensor of a perfect fluid may be formally identical to that of a viscous magnetohydrodynamic fluid. Here, we shall exploit the formal equivalence of (1) and (5), in which, for simplicity, we shall assume that no electromagnetic field is present (i.e., $E_{\mu\gamma}=0$). Parenthetically, we note that as a by-product of the analysis outlined below we will have shown that positive-curvature FRW models may be interpreted as viscous-fluid solutions—the solutions given below are among the first that exhibit this behavior.

Therefore, we are looking for an interior solution in which the material composition is that of a viscous fluid, but whose gravitational field (geometrical part) corresponds to that of the $k=+1$ FRW model represented by Eqs. (2), (3), and (4).

Let us assume that the four-velocity is of the form

$$u_\mu=[-\alpha,\beta R(1-r^2)^{-1/2},0,0]\ , \tag{6}$$

where $\alpha^2-\beta^2=1$, $\alpha\geq 1$, and α is a function that may depend on t and the radial coordinate r, and, since $u_\mu q^\mu=0$, let us assume that the heat-flux vector is of the form

$$q_\mu=Q[\beta,-\alpha R(1-r^2)^{-1/2},0,0]\ , \tag{7}$$

where $Q^2\equiv q_\mu q^\mu$. Then, the Einstein field equations $G_{\mu\gamma}=M_{\mu\gamma}$, for the metric given by (2) and (3), reduce to

$$\begin{aligned}
&3CR^{-3}=\rho\alpha^2+p\beta^2-\tfrac{4}{3}\eta\beta^2X-2Q\alpha\beta\ ,\\
&0=\rho\beta^2+p\alpha^2-\tfrac{4}{3}\eta\alpha^2X-2Q\alpha\beta\ ,\\
&0=p+\tfrac{2}{3}\eta X\ ,\\
&0=\rho+p-\tfrac{4}{3}\eta X-Q(\alpha^2+\beta^2)(\alpha\beta)^{-1}\ ,
\end{aligned} \tag{8}$$

where

$$X=\dot\alpha+\beta' R^{-1}(1-r^2)^{1/2}-\beta R^{-1}r^{-1}(1-r^2)^{1/2}\ , \tag{9}$$

and differentiation with respect to t and r is denoted, respectively, by a dot and by a prime.

Equations (8) have the solution

$$\begin{aligned}
&\rho=3\alpha^2CR^{-3},\quad p=\beta^2CR^{-3}\\
&Q=3\alpha\beta CR^{-3},\quad \eta X=-\tfrac{3}{2}\beta^2CR^{-3}\ .
\end{aligned} \tag{10}$$

The solution satisfies all the appropriate energy conditions (Hawking and Ellis[7]) in which the density and pressure are always non-negative. Equation (10) represents a physically acceptable solution provided $\eta\geq 0$, i.e., $X\leq 0$, viz.,

$$\dot\alpha+\beta' R^{-1}(1-r^2)^{1/2}-\beta R^{-1}r^{-1}(1-r^2)^{1/2}\leq 0\ . \tag{11}$$

Equations (10), with an α satisfying (11), represent a class of viscous-fluid solutions of Einstein's equations in which the geometrical part is identical to that of the positive-curvature FRW model represented by (2). Further specification of the solution entails the specification of the function α. Many choices exist for α [subject to (11)] but we shall attempt to find an α appropriate to the matchup problem under investigation. This specification essentially amounts to finding an appropriate set of boundary conditions for α.

We want the viscous-fluid solution to be valid for $0<r\leq r_S$, where r_S is the surface of the star (i.e., in the interior of the star). (Note that we could impose the weaker domain of validity $r_c\leq r\leq r_S$.) The model we have in mind is one in which the outer layers of the star are essentially dust; that is, all the viscous effects present

play their crucial role toward the innermost regions of the star—in the outer regions we simply have dust. This is, of course, physically reasonable, and as we shall see, of great practical use. Thus we impose the boundary condition that

$$\alpha=1 \quad \text{at } r=r_S \ , \tag{12}$$

so that at $r=r_S$ the fluid becomes dust and the model is identical to the standard underlying perfect-fluid model.

Alternatively, we could have chosen the boundary condition $\alpha=1$ at $r=r_v$ where $r_c \le r_v \le r_S$, so that at $r=r_v$ the viscous-fluid quantities reduce to their perfect-fluid counterparts. We could then smoothly match the viscous-fluid interior ($r_c \le r \le r_v$) at $r=r_v$ to the standard dust solution valid in the region $r_v \le r \le r_S$ (which then matches in the usual way to the Schwarzschild exterior at $r=r_S$). Moreover, perhaps the above prescription is more in keeping with the representation of the outer layers as dust. However, for convenience, in the remainder of this article we will consider the extreme case $r_v=r_S$ in which the thickness of the outer dust layer shrinks to 0 (i.e., $r_S-r_v\to 0$). Thus the matter constituting the star is only formally equivalent to dust at the surface of the star $r=r_S$. (Clearly the analysis that follows can be trivially generalized to the case in which $r_S-r_v\neq 0$.)

We note that with the boundary condition (12), $\beta=0$ and $p=Q=0$ at the surface of the star. In particular, at $r=r_S$ the matter is comoving. Consequently, the surface of the star will be at $r=r_S=$a constant in the coordinate system being used (since at $r=r_S$ the matter is comoving in this coordinate system). This makes the matchup with the exterior region straightforward (and identical to the matchup in the standard model) and avoids the problems outlined in the Introduction. Indeed, if (12) is satisfied, again choosing

$$1=\frac{M}{4\pi {r_S}^3} \tag{13}$$

ensures that the model matches up continuously to a Schwarzschild exterior.

We also wish to impose some conditions on the model as we approach $r=0$. As we have mentioned previously, we wish the viscous effects to play an increasingly important role as we approach the central regions; consequently we desire α and $|\beta|$ to increase as $r\to 0$. Therefore, we also impose the phenomenological boundary condition

$$\alpha\to\infty \quad \text{as } r\to 0 \tag{14}$$

(alternatively we could impose the boundary condition $\alpha\to\text{constant}>1$ as $r\to 0$).

To complete the solution we must therefore specify an α that satisfies inequality (11) and the boundary conditions (12) and (14) such that the solution is valid for $0\le r_c < r \le r_S < 1$ (at least in the collapsing phase $\pi\le\psi\le 2\pi$). It is more convenient at this point to work with β. Inequality (11) is satisfied if we impose the more restrictive conditions

$$\dot\beta\le 0,\ \ \beta'\le 0,\ \ \beta\ge 0 \ , \tag{15}$$

and the boundary conditions become (in terms of β)

$$\beta=0 \quad \text{at } r=r_S$$

and

$$\left.\begin{array}{l}\beta\to\text{positive constant}\\ \text{or}\\ \beta\to\infty\end{array}\right\}\text{as } r\to 0 \ . \tag{16}$$

There are many choices for β that satisfy (15) and (16). Let us put

$$\beta=f(t)g(r,r_S) \ . \tag{17}$$

If we investigate the t dependence of β, we simply want $\dot f\le 0$ (assuming f and g are positive functions). Possible choices for f might be (i) $f(t)=t^{-a}$ $(a>0)$, (ii) $f(t)=F(R(t))$ where $dF/dR\ge 0$ (since $\dot R$ is negative in the collapsing stage and $\dot f\le 0 \Longleftrightarrow (dF/dR)\dot R\le 0$), (iii) there exist solutions in which β is independent of t, i.e., $f(t)=\text{constant}$.

Let us now investigate the spatial dependence of β. If we choose

$$g(r,r_S)=\left[\frac{r_S}{r}-1\right]^b,\ \ b>0 \ , \tag{18}$$

we note that

$$g'=\frac{-br_S}{r^2}\left[\frac{r_S}{r}-1\right]^{b-1}$$

which is always negative so that inequality (11) is always satisfied. At $r=r_S$, $\beta=0$, and as $r\to 0$, $\beta\to\infty$. The solution is valid as $r\to 0$. Thus (18) is an acceptable choice for g.

Alternatively, we might choose

$$g(r,r_S)=a(r_S-r)^b,\ \ a,b>0 \ . \tag{19}$$

Again g' is negative so that inequality (11) is satisfied. When $r=r_S$, $\beta=0$, but as $r\to 0$, $g(r,r_S)\to a{r_S}^b$, a constant. The solution is valid for $0<r\le r_S$.

CONCLUSION

As an illustration, let us consider one particular model. Let us choose

$$\beta(r,t)=t^{-1}\left[\frac{r_S}{r}-1\right]^2 \ . \tag{20}$$

With this choice of β, the solution (10) becomes

$$\begin{aligned}
\rho&=3C\left[1+t^{-2}\left[\frac{r_S}{r}-1\right]^4\right]R^{-3} \ ,\\
p&=Ct^{-2}\left[\frac{r_S}{r}-1\right]^4 R^{-3} \ ,\\
Q&=3Ct^{-1}\left[\frac{r_S}{r}-1\right]^2\left[1+t^{-2}\left[\frac{r_S}{r}-1\right]^4\right]^{1/2}R^{-3} \ ,\\
\eta X&=-\tfrac{3}{2}Ct^{-2}\left[\frac{r_S}{r}-1\right]^4 R^{-3} \ ,
\end{aligned} \tag{21}$$

where $R(t)$ is given by (3) and the solution is valid for $0<r\le r_S$ and $\pi\le\psi\le 2\pi$. This is a solution of the viscous-fluid field equations in which the geometrical part is formally identical to the positive-curvature FRW line element (2).

Equations (21) represent the interior solution of a collapsing-viscous-fluid matter distribution. It is physically acceptable because $\rho\ge 0$, $p\ge 0$, and $\eta\ge 0$ since inequality (11) is satisfied. As $r\to 0$, $\beta\to\infty$ and we note that $\rho/p\to 3$, so that at the star's center the matter is in a radiationlike state, which is an advantage of the model. The interior solution represents a collapsing-viscous-fluid ball since $\dot{R}<0$ and at $t=\pi R_0$, $R=0$. If we examine the density on a spatial hypersurface as $t\to\pi R_0$ (i.e., $R\to 0$) we find that it increases with time, possibly indicating the motion of matter toward the center.

At $r=r_S$, $\beta=0$ and the matter distribution becomes that of dust and the matter is comoving relative to the coordinate system being used. Choosing $1=M/4\pi r_S{}^3$, the gravitational field matches continuously at $r=r_S$ to an exterior Schwarzschild gravitational field as in the standard case. Thus the complete solution represents a collapsing viscous fluid matched continuously across its boundary to an exterior Schwarzschild vacuum spacetime.

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