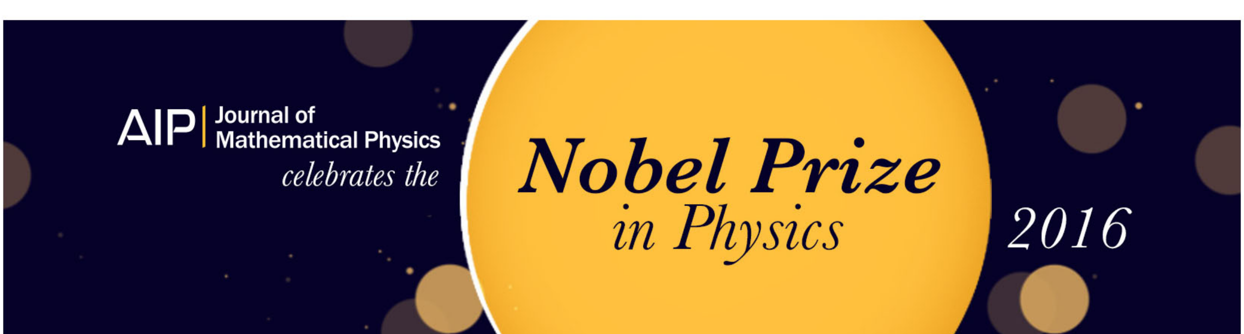
AIP | Journal of Mathematical Physics

Affine conformal vectors in space-time

A. A. Coley and B. O. J. Tupper

Citation: Journal of Mathematical Physics **33**, 1754 (1992); doi: 10.1063/1.529652
View online: http://dx.doi.org/10.1063/1.529652
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/33/5?ver=pdfcov
Published by the AIP Publishing



Affine conformal vectors in space-time

A. A. Coley
Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada

B. O. J. Tupper
Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick, E3B 5A3, Canada

(Received 19 September 1991; accepted for publication 30 December 1991)

All space-times admitting a proper affine conformal vector (ACV) are found. By using a theorem of Hall and da Costa, it is shown that such space-times either (i) admit a covariantly constant vector (timelike, spacelike, or null) and the ACV is the sum of a proper affine vector and a conformal Killing vector or (ii) the space-time is $2+2$ decomposable, in which case it is shown that no ACV can exist (unless the space-time decomposes further). Furthermore, it is proved that all space-times admitting an ACV and a null covariantly constant vector (which are necessarily generalized *pp*-wave space-times) must have Ricci tensor of Segré type {2,(1,1)}. It follows that, among space-times admitting proper ACV, the Einstein static universe is the only perfect fluid space-time, there are no non-null Einstein–Maxwell space-times, and only the *pp*-wave space-times are representative of null Einstein–Maxwell solutions. Otherwise, the space-times can represent anisotropic fluids and viscous heat-conducting fluids, but only with restricted equations of state in each case.

I. INTRODUCTION

There has been much recent interest in the existence of symmetries in the space-times manifold of general relativity and in the corresponding symmetry vector fields ξ^a. These symmetries are often discussed in the context of fluid (perfect, anisotropic, or viscous heat-conducting) space-times and the effect of these symmetries on the kinematic and dynamic properties of the fluids is studied.

In general, these symmetries may be characterized by the Lie derivative of the metric tensor g_{ab} along ξ^a, which can be written in the form

$$\underset{\xi}{\mathscr{L}}\, g_{ab}=2\psi g_{ab}+K_{ab}, \tag{1.1}$$

where $\psi=\psi(x_a)$ is a scalar function and K_{ab} is a symmetric tensor not proportional to g_{ab}. When $K_{ab}=0$, the various special symmetries of the form (1.1) are Killing vectors (KV), with $\psi=0$, homothetic vectors (HV), with $\psi=\text{const}\neq 0$, special conformal Killing vectors (SCKV), with $\psi_{;ab}=0$, $\psi_{,a}\neq 0$, and proper conformal Killing vectors (CKV), with $\psi_{;ab}\neq 0$. (Note that the term "proper CKV" is usually used to exclude KV and HV; our use of the term also excludes SCKV.) When K_{ab} is not zero and is covariantly constant, i.e., $K_{ab;c}=0$, the corresponding symmetries are affine vectors (AV), with $\psi_{,a}=0$, proper affine conformal vectors (ACV), with $\psi_{;ab}\neq 0$, and special affine conformal vectors (SACV), with $\psi_{;ab}=0$, $\psi\neq 0$.

All space-times admitting SCKV have been found by the present authors,[1] who have also shown that these are the only space-times admitting SACV.[2] There are very few such space-times, none of which can represent a perfect fluid or an Einstein–Maxwell field (other than the *pp*-wave null field), but they each can represent an anisotropic fluid with a particularly restrictive equation of state or a viscous heat-conducting fluid. Hall and da Costa[3] have made a thorough study of AVs. The purpose of the present investigation is to complete the study of the last four of the above symmetries by considering the case of ACVs.

An ACV or a CKV (which may be regarded as a special case of an ACV in which K_{ab} is proportional to g_{ab}) generates a conformal collineation characterized by

$$\underset{\xi}{\mathscr{L}}\left\{{}^{\,a}_{bc}\right\}=\delta^a_b\psi_{,c}+\delta^a_c\psi_{,b}-g_{bc}\psi^{,a}_{,}. \tag{1.2}$$

Conformal collineations and, in particular, ACVs have been the subject of a number of studies;[4–8] the problem of the existence of ACVs for positive definite manifolds has been solved completely,[9] but no complete solution of the problem exists for space-times. In this article we shall present the complete solution by finding all space-times that admit ACV. In particular, we shall show that the Einstein static universe, which is known to admit an ACV,[6] is the only perfect fluid space-time and also the only FRW model to do so; this is contrary to an assertion made in Ref. 8.

 0022-2488/92/051754-11$03.00

The existence of the covariantly constant tensor K_{ab} imposes strong restrictions on the possible space-times. It has been shown[3] that if a simply connected space-time admits a global, nowhere zero, covariantly constant second-order symmetric tensor, then one of the following three possibilities must occur.

(a) There exists locally a timelike or spacelike nowhere zero covariantly constant vector field $\eta_a=\eta_{,a}$, such that $K_{ab}=\eta_{,a}\eta_{,b}$ and the space-time is locally decomposable into a $1+3$ space-time.

(b) There exists locally a null nowhere zero covariantly constant vector field $\eta_a\equiv\eta_{,a}$, such that $K_{ab}\equiv\eta_{,a}\eta_{,b}$ and the space-time is the generalized *pp*-wave space-time,[1,10] which, in general, is not decomposable.

(c) The space-time is locally decomposable into a $2+2$ space-time and no covariantly constant vector exists unless the space-time decomposes further. If the space-time does decompose into a $1+1+2$ space-time (which also can be regarded as a special case of a $1+3$ space-time), there exist two covariantly constant vectors $\eta_a\equiv\eta_{,a}$ and $\lambda_a\equiv\lambda_{,a}$.

In cases (a) and (b), Eq. (1.1) becomes

$$\xi_{a;b}+\xi_{b;a}=2\psi g_{ab}+\eta_{,a}\eta_{,b}, \tag{1.3}$$

which, since $\eta_{;ab}=0$, can be written in the form

$$(\xi_a-\tfrac{1}{2}\eta\eta_{,a})_{;b}+(\xi_b-\tfrac{1}{2}\eta\eta_{,b})_{;a}=2\psi g_{ab}, \tag{1.4}$$

so that

$$\xi_a=\zeta_a+\tau_a, \tag{1.5}$$

where ζ_a is a CKV; i.e.,

$$\zeta_{a;b}+\zeta_{b;a}=2\psi g_{ab} \tag{1.6}$$

and $\tau_a=\frac{1}{2}\eta\eta_{,a}$ is an AV. Thus, in cases (a) and (b), the ACV is the sum of a CKV and an AV; the problem in these cases becomes one of finding space-times admitting both a CKV and a covariantly constant tensor. Note that in these cases K_{ab} is generally given by $K_{ab}=2\nu g_{ab}+\eta_{,a}\eta_{,b}$, where ν is a constant.[3] However, an analysis similar to that above shows that the AV is then the sum of an HV and a proper AV, and this can hold only if the space-time admits an HV; if no HV exists, then ν must be zero. (This fact is of particular relevance in Sec. VI.) Since we are interested only in proper ACV, i.e., ACV for which additive constants in ψ, corresponding to HV, are factored out, we can always take $\nu=0$ even when an HV does exist.

In Secs. II–IV of this paper we discuss, respectively, the cases when the covariantly constant vector is timelike, spacelike, and null, while in Sec. V, it is shown that no ACVs exist in the case of a $2+2$ decomposable space-time. Concluding remarks are made in Sec. VI and the Appendix contains the proof of a result that is required in Secs. II and III. Throughout this investigation we are interested only in fluid space-times that admit *proper* ACV, so we shall discard the case in which $\psi_{;ab}=0$, i.e., in which ξ^a is a SACV.

Finally, we note for future reference the expressions for the energy-momentum tensors corresponding to an anisotropic fluid and to a viscous heat-conducting fluid, respectively. The first of these is

$$T_{ab}=\mu u_a u_b+p_{\|}n_a n_b+p_{\perp}p_{ab} \tag{1.7}$$

and the second is

$$T_{ab}=\mu u_a u_b+ph_{ab}-2\eta\sigma_{ab}+q_a u_b+q_b u_a, \tag{1.8}$$

where μ is the energy density, p is the isotropic pressure, $h_{ab}=g_{ab}+u_a u_b$ is the projection tensor onto the hypersurface orthogonal to the four velocity u^a, q^a is the heat conduction vector satisfying $q^a u_a=0$, $\eta(\geqslant 0)$ is the shear viscosity coefficient, σ_{ab} is the tensor, n^a is a spacelike unit vector orthogonal to u^a, p_{ab} is the projection tensor onto the two-plane orthogonal to u^a and n^a, and $p_{\|}$, $p_{\perp}$ denote pressure parallel to and perpendicular to n^a, respectively.

II. THE TIMELIKE CASE

Consider the case in which the covariantly constant vector $\eta_{,a}$ is timelike. Choosing coordinates such that $\eta=-t(\equiv -x^0)$, i.e., $\eta_{,a}=-\delta^0_a$, the metric takes the form

$$ds^2=-dt^2+g_{\alpha\beta}(x^\gamma)dx^\alpha\,dx^\beta, \tag{2.1}$$

where the Greek suffixes take the values 1, 2, 3. We need to find the CKV, if any, admitted by this space-time. The CKV equations (1.6) for the metric (2.1) are

$$\zeta_{0,0}=-\psi, \tag{2.2}$$

$$\zeta_{0,\alpha}+\zeta_{\alpha,0}=0, \tag{2.3}$$

$$\zeta_{\alpha;\beta}+\zeta_{\beta;\alpha}=2\psi g_{\alpha\beta}. \tag{2.4}$$

Putting $\psi=\Sigma_{,00}$, where $\Sigma=\Sigma(t,x^\alpha)$, and integrating Eq. (2.2), we obtain

$$\zeta_0=-\Sigma_{,0}+A(x^\alpha). \tag{2.5}$$

Equation (2.3) then yields

$$\zeta_\alpha=\Sigma_{,\alpha}-A_{,\alpha}t+B_\alpha(x^\gamma), \tag{2.6}$$

and Eq. (2.4) becomes

$$2\Sigma_{;\alpha\beta}-2A_{;\alpha\beta}t+B_{\alpha;\beta}+B_{\beta;\alpha}=2\psi g_{\alpha\beta}. \tag{2.7}$$

Differentiating twice with respect to (wrt)t, we obtain

$$\Sigma_{;\alpha\beta,00} = \psi_{,00} g_{\alpha\beta};$$

i.e.,

$$\psi_{;\alpha\beta} = \psi_{,00} g_{\alpha\beta}, \tag{2.8}$$

which is Eq. (A10), with $\epsilon = -1$. Identifying x^1, x^2 of the Appendix, with t, x, Eqs. (A24)–(A26) show that the possible metrics are

$$ds^2 = -dt^2 + dx^2 + \sin^2 kx f^2(y,z)(dy^2 + dz^2), \tag{2.9}$$

$$ds^2 = -dt^2 + dx^2 + \cosh^2 kx f^2(y,z)(dy^2 + dz^2), \tag{2.10}$$

$$ds^2 = -dt^2 + dx^2 + \sinh^2 kx f^2(y,z)(dy^2 + dz^2), \tag{2.11}$$

$$ds^2 = -dt^2 + dx^2 + x^2 f^2(y,z)(dy^2 + dz^2). \tag{2.12}$$

The metric (2.12), corresponding to $\alpha = 0$ in the Appendix, is one of the SACV space-times.[2] In this case it is easily shown that, in order to satisfy all components of Eq. (2.7), the constant β in Eqs. (A29) and (A31) must be zero so that ψ is a linear function of t, as shown in Ref. 2. This implies that the space-times cannot admit any ACVs other than the SACV. The space-times (2.10) and (2.11) are listed for completeness but, since neither of them can satisfy the dominant energy condition (DEC), we shall exclude them from further discussion. Thus the metric (2.9) represents the only space-time that locally admits a proper ACV with timelike covariantly constant vector and that can also satisfy the DEC. From Eqs. (2.2)–(2.4) and (A15), (A24), (A27), and (A30), the expressions for ψ and the ACV ξ^a corresponding to this space-time are

$$\xi^a = (k^{-2} \cos kx \sin kt - \tfrac{1}{2}t, k^{-2} \sin kx \cos kt, 0, 0),$$
$$\psi = k^{-1} \cos kx \cos kt. \tag{2.13}$$

Note that, while the ACV given by Eq. (2.13) always exists for the metric (2.9), in special cases, i.e., for particular metric functions $f^2(y,z)$, additional CKV, and thus additional ACV, may exist. For example, the metric (2.9) includes the Einstein static universe which admits eight proper CKV, each one of which can be combined with the AV to produce an ACV.

The space-time (2.9) satisfies then DEC if

$$\Delta_f \leqslant -k^2 \cos 2kx, \tag{2.14}$$

where $\Delta_f \equiv \Delta_f(y,z)$ is defined by

$$\Delta_f = f^{-4}(f f_{yy} + f f_{zz} - f_y^2 - f_z^2). \tag{2.15}$$

Note that the condition (2.14) is always satisfied if $\Delta_f \leqslant -k^2$.

The nonzero Einstein tensor components for the metric are

$$G_{00} = 2k^2 - k^2 \cot^2 kx - \Delta_f \csc^2 kx,$$
$$G_1^1 = k^2 \cot^2 kx + \Delta_f \csc^2 kx, \tag{2.16}$$
$$G_2^2 = G_3^3 = -k^2,$$

and these satisfy the Einstein field equations for a perfect fluid if and only if $\Delta_f = -k^2$, i.e., the two-space $f^2(y,z)(dy^2 + dz^2)$ must be a space of constant negative curvature. In this case, we can choose coordinates such that $f = \operatorname{sech} y$ and the space-time is the Einstein static universe which, without cosmological constant, satisfies the equation of state $\mu + 3p = 0$.

From Eq. (2.16), the space-time (2.9) cannot represent an Einstein–Maxwell field, but can represent an anisotropic fluid with $p_\perp = -k^2$, or a viscous heat-conducting fluid with $\mu + 3p = 0$, as in the case of SACV.[2]

III. THE SPACELIKE CASE

When the covariantly constant vector $\eta_{,a}$ is spacelike, we may choose coordinates such that $\eta = z(\equiv x^3)$; i.e., $\eta_{,a} = \delta_a^3$, and the metric takes the form

$$ds^2 = dz^2 + g_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta, \tag{3.1}$$

where now the Greek suffixes take the values 0,1,2. The CKV equations (1.5) are

$$\zeta_{3,3} = \psi, \tag{3.2}$$

$$\zeta_{3,\alpha} + \zeta_{\alpha,3} = 0, \tag{3.3}$$

$$\zeta_{\alpha;\beta} + \zeta_{\beta;\alpha} = 2\psi g_{\alpha\beta}. \tag{3.4}$$

Putting $\psi = \Xi_{,33}$, where $\Xi = \Xi(t, x^\alpha)$, and following the corresponding process of Sec. II, we obtain in place of Eqs. (2.5)–(2.8) the following equations:

$$\zeta_3 = \Xi_{,3} + C(x^\alpha), \tag{3.5}$$

$$\zeta_\alpha = -\Xi_{,\alpha} - C_{,\alpha} z + D_\alpha(x^\gamma), \tag{3.6}$$

$$-2\Xi_{;\alpha\beta} - 2C_{;\alpha\beta} z + D_{\alpha;\beta} + D_{\beta;\alpha} = 2\psi_{;\alpha\beta}, \tag{3.7}$$

$$\psi_{;\alpha\beta} = -\psi_{,33} g_{\alpha\beta}, \tag{3.8}$$

the last of which is Eq. (A10) with $\epsilon = +1$. We identify x^1 of the Appendix with z, but x^2 of the Appendix can be identified either (i) with t, or (ii) with y (equivalently with x). The possible metrics in each case are

$$\text{(i)} \quad ds^2 = dz^2 - dt^2 + a_t^2(t) g^2(x,y)(dx^2 + dy^2), \tag{3.9}$$

with $a_t = \sin kt$, $\cosh kt$, $\sinh kt$, or t, and

$$\text{(ii)} \quad ds^2 = dz^2 + dy^2 + a_y^2(y)h^2(t,x)(-dt^2 + dx^2), \tag{3.10}$$

with $a_y = \sin ky$, $\cosh ky$, $\sinh ky$, or y.

As in the case of the space-time with metric (2.12), case (i) with $a_t = t$ and case (ii) with $a_y = y$ admit only SACV,[2] while case (i) with $a_t = \cosh kt$ or $\sinh kt$ and all of the space-times of case (ii) cannot satisfy the DEC and so will not be considered further. Hence the metric (3.9) with $a_t = \sin kt$, i.e.,

$$ds^2 = -dt^2 + \sin^2 kt g^2(x,y)(dx^2 + dy^2) + dz^2 \tag{3.11}$$

represents the only space-time which locally admits a proper ACV with spacelike covariantly constant vector and which also can satisfy the DEC. From Eqs. (3.2)–(3.4) and(A15), (A24), (A27), and (A30), the expressions for the ACV ξ^a and for ψ corresponding to this space-time are

$$\xi^a = (k^{-2} \cos kz \sin kt, 0, 0, k^{-2} \sin kz \cos kt + \tfrac{1}{2}z),$$
$$\psi = k^{-1} \cos kz \cos kt. \tag{3.12}$$

The space-time (3.11) satisfies the DEC if

$$\Delta_g \leq k^2 \cos 2kt, \tag{3.13}$$

which is always true if $\Delta_g \leq -k^2$, where Δ_g is the expression for $g(x,y)$ corresponding to Eq. (2.15). The space-time cannot represent a perfect fluid or an Einstein–Maxwell field but can represent an anisotropic fluid satisfying $\mu + p_{\parallel} - 2p_{\perp} = 0$ or a viscous heat-conducting fluid.

IV. THE NULL CASE

In the case when $\eta_{,a}$ is null, since it is a gradient vector and a null KV, it follows that we have a generalized *pp*-wave space-time[10] with metric of the form

$$ds^2 = P^{-2}(dx^2 + dy^2) - 2\,du(dv - m\,dx + H\,du), \tag{4.1}$$

where H, m, and P are arbitrary functions of u, x, and y only. Labeling the coordinates $(u,v,x,y) \equiv (x^0,x^1,x^2,x^3)$, the null KV $k^a = \eta_,{}^a$ is given by $k^a = \delta_1^a$, i.e., $k_a = -\delta_a^0$, so that

$$\eta = -u. \tag{4.2}$$

When the Ricci scalar vanishes, the metric (4.1) can be transformed into

$$ds^2 = dx^2 + dy^2 - 2\,du\,dv - 2H\,du^2; \tag{4.3}$$

i.e., the metric (4.1) with $P = 1$ and $m = 0$, and Eq. (4.2) still holds.

We require those members of the two sets of metrics (4.1) and (4.3) which admit CKV; we consider first the metric (4.1), i.e., the case $R \neq 0$. The nonzero components of the Ricci tensor for this metric are

$$\begin{aligned}
R_{00} &= P^2(H_{xx} + H_{yy} + m_{ux} + \tfrac{1}{2}m_y^2 P^2) \\
&\quad + 2P^{-2}(PP_{uu} - 2P_u^2), \\
R_{02} &= -\tfrac{1}{2}m_{yy}P^2 - m_y PP_y + P^{-2}(PP_{ux} - P_u P_x), \\
R_{03} &= \tfrac{1}{2}m_{xy}P^2 + m_y PP_x + P^{-2}(PP_{uy} - P_u P_y), \\
R_{22} &= R_{33} = \tfrac{1}{2}P^{-2}R = P^{-2}(PP_{xx} + PP_{yy} - P_x^2 - P_y^2),
\end{aligned} \tag{4.4}$$

and the CKV equations (1.5) are

$$\begin{aligned}
\zeta_{0,0} &= (H_u + mm_u P^2 + mH_x P^2)\zeta_1 + (m_u + H_x)P^2\zeta_2 \\
&\quad + H_y P^2 \zeta_3 - 2H\psi, \\
\zeta_{0,1} + \zeta_{1,0} &= -2\psi, \\
\zeta_{0,2} + \zeta_{2,0} &= 2(H_x - mP^{-1}P_u)\zeta_1 - 2P^{-1}P_u\zeta_2 \\
&\quad - m_y P^2 \zeta_3 + 2m\psi, \\
\zeta_{0,3} + \zeta_{3,0} &= 2\left(H_y + \frac{1}{2}m_y m P^2\right)\zeta_1 + m_y P^2 \zeta_2 \\
&\quad - 2P^{-1}P_u\zeta_3, \\
\zeta_{1,1} &= \zeta_{1,2} + \zeta_{2,1} = \zeta_{1,3} + \zeta_{3,1} = 0, \\
\zeta_{2,2} &= -(P^{-3}P_u + mP^{-1}P_x + m_x)\zeta_1 - P^{-1}P_x\zeta_2 \\
&\quad + P^{-1}P_y\zeta_3 + P^{-2}\psi, \\
\zeta_{2,3} + \zeta_{3,2} &= -(m_y + 2mP^{-1}P_y)\zeta_1 - 2P^{-1}P_y\zeta_2 \\
&\quad - 2P^{-1}P_x\zeta_3, \\
\zeta_{3,3} &= -(P^{-3}P_u - mP^{-1}P_x)\zeta_1 + P^{-1}P_x\zeta_2 \\
&\quad - P^{-1}P_y\zeta_3 + P^{-2}\psi.
\end{aligned} \tag{4.5}$$

From the (1,1) component of Eq. (4.5) we obtain

$$\zeta_1 = A(u,x,y), \tag{4.6}$$

and the (1,2) and (1,3) components then imply that

$$\zeta_2 = -A_x v + B(u,x,y), \tag{4.7}$$

$$\zeta_3 = -A_y v + C(u,x,y). \tag{4.8}$$

Now for a CKV, the Lie derivative of the Ricci tensor satisfies[1]

$$\underset{\zeta}{\mathscr{L}} R_{ab} = -2\psi_{;ab} - g_{ab}\Box\psi, \tag{4.9}$$

where $\Box\psi \equiv g^{ab}\psi_{;ab}$, and (1,1), (1,2), and (1,3) components of this expression lead to

$$\psi_{vv} = 0, \tag{4.10}$$

$$\psi_{vx} = \tfrac{1}{4}RA_x, \quad \psi_{vy} = \tfrac{1}{4}RA_y. \tag{4.11}$$

Using Eqs. (4.6)–(4.8) the (0,0) and (0,1) components of Eq. (4.5) become

$$\zeta_{0,0} = (H_u + mm_u P^2 + mH_x P^2)A + (m_u + H_x)P^2 \times(-A_x v + B) + H_y P^2(-A_y v + C) - 2H\psi, \tag{4.12}$$

$$\zeta_{0,1} = -A_u = 2\psi, \tag{4.13}$$

and the integrability condition $\zeta_{0,01} = \zeta_{0,10}$ leads to

$$A_{uu} + 2\psi_n = (m_u + H_\alpha)P^2 A_x + H_y P^2 A_y + 2H\psi_x. \tag{4.14}$$

Differentiating wrt v and using Eq. (4.10) results in

$$\psi_{uv} = 0. \tag{4.15}$$

Similarly, Eq. (4.13), the (0,2) and (0,3) components of Eq. (4.5), and the integrability conditions $\zeta_{0,21} = \zeta_{0,12}$ and $\zeta_{0,31} = \zeta_{0,13}$ lead to

$$A_{ux} + 2\psi_x + 2P^{-1}P_u A_x + m_y P^2 A_y + 2m\psi_v = 0, \tag{4.16}$$

$$A_{uy} + 2\psi_y + 2P^{-1}P_u A_y + m_y P^2 A_x = 0. \tag{4.17}$$

Further differentiation wrt v yields

$$\psi_{vx} = \psi_{vy} = 0 \tag{4.18}$$

and this, combined with Eq. (4.11), and the fact that $R \neq 0$, implies that

$$A_x = A_y = 0; \tag{4.19}$$

i.e., $A = A(u)$ and so ζ_1, ζ_2, ζ_3 are each independent of v; i.e.,

$$\zeta_1 = A(u), \quad \zeta_2 = B(u,x,y), \quad \zeta_3 = C(u,x,y). \tag{4.20}$$

Differentiating the (2.2) component on Eq. (4.5) wrt v and using Eq. (4.20) we obtain

$$\psi_v = 0, \tag{4.21}$$

and using this and Eq. (4.19) we obtain from Eqs. (4.16) and (4.17)

$$\psi_x = \psi_y = 0; \tag{4.22}$$

i.e., $\psi = \psi(u)$, and from Eq. (4.14), $\psi_u = -\frac{1}{2}A_{uu}$; i.e.,

$$\psi = \tfrac{1}{2}(\alpha - A_u), \tag{4.23}$$

where α is a constant and, consequently, $\Box\psi = 0$.

Equation (4.13) now integrates to give

$$\zeta_0 = -\alpha v + D(u,x,y), \tag{4.24}$$

so that, when $R \neq 0$, the CKV is of the form

$$\zeta^a = [-A(u), \alpha v - D(u,x,y) + (2H + m^2P^2)A(u) + mP^2 B(u,x,y), mP^2 A(u) + P^2 B(u,x,y), P^2 C(u,x,y)], \tag{4.25}$$

where the functions A, B, C, and D satisfy the following set of differential equations:

$$\begin{aligned}
D_u &= (H_u + mm_u P^2 + mH_x P^2)A + (m_u + H_x)P^2 B + H_y P^2 C - H(\alpha - A_u),\\
B_x &= -(P^{-3}P_u + mP^{-1}P_x + m_x)A - P^{-1}P_x B + P^{-1}P_y C + \tfrac{1}{2}P^{-2}(\alpha - A_u),\\
C_y &= -(P^{-3}P_u - mP^{-1}P_x)A + P^{-1}P_x B - P^{-1}P_y C + \tfrac{1}{2}P^{-2}(\alpha - A_u),\\
D_x + B_u &= 2(H_x - mP^{-1}P_u)A - 2P^{-1}P_u B - m_y P^2 C + m(\alpha - A_u),\\
D_y + C_u &= 2(H_y + \tfrac{1}{2}m_y m P^2)A + m_y P^2 B - 2P^{-1}P_u C,\\
B_y + C_x &= -(m_y + 2mP^{-1}P_y)A - 2P^{-1}P_y B - 2P^{-1}P_x C.
\end{aligned} \tag{4.26}$$

Elimination of α, A, B, C, and D from these equations will result in an expression connecting H, m, P and their derivatives which specifies those members of the general set of space-times with metric (4.1) which admit CKV. Note that when A is a quadratic function of u, we have $\psi_{uu} = 0$; i.e., $\psi_{;ab} = 0$, so that ζ^a is a SCKV; this case should be excluded.

The space-time with metric (4.1) has an energy-momentum tensor that can be written in the form[1]

$$T_{ab}=\tfrac{1}{2}R(k_al_b+k_bl_a)+Mk_ak_b, \tag{4.27}$$

where $k_a\equiv\eta_{,a}$ is the null KV, l_a is a null vector satisfying $k_al^a=1$, and M is a scalar function of the coordinates. When $M\neq 0$, T_{ab} is of Segré type {2,(1 1)}, while if $M=0$, T_{ab} is of Segré type {(1,1)(1 1)} where, in each case, the two bracketed spacelike eigenvectors have zero eigenvalue. Note that the metric (4.3), for which $R=0$, always has T_{ab} of Segré type {2,(1 1)}.

Using Eq. (4.27) with $M=0$, the expression (4.4) for R_{ab}, and the fact that $k^a=\delta^a_1$, it is easily shown that T_{ab} for the metric (4.1) will be of Segré type {(1,1)(1 1)} if and only if the second null eigenvector l^a is given by

$$l^a=(-1,-R^{-1}R_{00}+2mP^2R^{-1}R_{02}+H,\; 2P^2R^{-1}R_{02},2P^2R^{-1}R_{03}), \tag{4.28}$$

whence the condition $l_al^a=0$ yields

$$2P^2[(R_{02})^2+(R_{03})^2]-RR_{00}=0. \tag{4.29}$$

Now the left-hand side of this expression is a scalar quantity $(\tfrac{1}{2}R^2l_al^a)$, so that

$$\underset{\zeta}{\mathscr{L}}(R^2l_al^a)=\zeta^b(R^2l_al^a)_{,b}=0. \tag{4.30}$$

Using Eqs. (4.5), (4.25), and (4.29), after a straightforward but tedious computation, Eq. (4.30) simplifies to $\psi_{uu}=0$, so that ζ^a is necessarily a SCKV and we have the following theorem.

Theorem 1: The generalized *pp*-wave space-times given by the metric (4.1) and which have energy-momentum tensor of Segré type {(1,1)(1 1)} cannot admit a proper CKV but may admit a SCKV.

Alternatively, we may say that a necessary, but not sufficient, condition for the space-time with metric (4.1) to admit a proper CKV is that the corresponding energy-momentum tensor be of Segré type {2,(1 1)}. Thus a proper ACV, ξ^a, may exist only for Segré type {2,(1 1)} and is then given by

$$\xi^a=\zeta^a-\tfrac{1}{2}u\delta^a_1, \tag{4.31}$$

where ζ^a is given by Eq. (4.25).

Turning to the case when $R=0$, i.e., to the metric (4.3), we note that, unlike the $R\neq 0$ case, comparison of Eqs. (4.11) and (4.18) does not lead to $A_x=A_y=0$ and, consequently, ψ is not necessarily a function of u only. Maartens and Maharaj[11] have investigated the CKV equations for the metric (4.3) and have shown that the general CKV, ζ^a, and the conformal scalar, ψ, are of the form

$$\zeta^0=\frac{1}{2}\alpha(x^2+y^2)+ax+by+c,$$

$$\zeta^1=\alpha v^2+(a_ux-b_uy-c_u+2d)v+F(u,x,y),$$

$$\zeta^2=(\alpha x+a)v+\tfrac{1}{2}a_ux^2+b_uxy-\tfrac{1}{2}a_uy^2+dx+ey+f, \tag{4.32}$$

$$\zeta^3=(\alpha y+b)v-\tfrac{1}{2}b_ux^2+a_uxy+\tfrac{1}{2}b_uy^2-ex+dy+g,$$

$$\psi=\alpha v+a_ux+b_uy+d, \tag{4.33}$$

where α is an arbitrary constant, a, b, c, d, e, f, and g are functions of u only, and the above expressions are subject to the following conditions:

$$(\alpha x+a)H_x+(\alpha y+b)H_y-2\alpha H+a_{uu}x+b_{uu}y-c_{uu}+2d_u=0, \tag{4.34}$$

$$F_u=2(d-c_u)H-[\tfrac{1}{2}\alpha(x^2+y^2)+ax+by+c]H_u-[\tfrac{1}{2}a_ux^2+b_uxy-\tfrac{1}{2}a_uy^2+dx+ey+f]H_x-[-\tfrac{1}{2}b_ux^2+a_uxy+\tfrac{1}{2}b_uy^2-ex+dy+g]H_y, \tag{4.35}$$

$$F_x=-2(\alpha x+a)H+\tfrac{1}{2}a_{uu}x^2+b_{uu}xy-\tfrac{1}{2}a_{uu}y^2+d_ux+e_uy+f_u, \tag{4.36}$$

$$F_y=-2(\alpha y+b)H-\tfrac{1}{2}b_{uu}x^2+a_{uu}xy+\tfrac{1}{2}b_{uu}y^2-e_ux+d_uy+g_u. \tag{4.37}$$

The three integrability conditions $F_{ux}=F_{xu}$, $F_{uy}=F_{yu}$, $F_{xy}=F_{yx}$, together with Eq. (4.34), form a set of four conditions on the metric function H and its derivatives which satisfy those space-times with general metric (4.3) that admit CKV. Note that the case $\alpha=a_u=b_u=d_{uu}=0$; i.e., $\psi\propto u$, should be discarded since it implies that the CKV is a SCKV.[1] The equations (4.32)–(4.37) are all given in Ref. 11, where there are also two examples, one of which (the conformally flat null Einstein–Maxwell solution) admits seven CKV and so admits seven ACV, showing that space-times exist that admit multiple ACV. The ACV is again given by Eq. (4.31) with ζ^a now given by Eq. (4.32).

Since the ACV space-times with metric (4.1) or (4.3) necessarily have T_{ab} of Segré type $\{2,(1\ 1)\}$, they cannot represent perfect fluids, anisotropic fluids of the form (1.6), or non-null Einstein–Maxwell fields. They can represent viscous heat-conducting fluids and, in the case $R=0$ only, null Einstein–Maxwell fields. In the viscous fluid case, if we write the heat conduction vector as $q^a=Qe^a$, where e^a is a unit spacelike vector orthogonal to u^a, and assume that e^a is an eigenvector of the shear tensor σ_{ab} with eigenvalue λ, then the viscous fluid interpretation of metric (4.1) must satisfy the conditions[12]

$$M=2Q=\mu+3p, \quad p=-\eta\lambda, \tag{4.38}$$

where M is the scalar quantity in Eq. (4.27). In the case of the metric (4.3), these conditions hold together with $\mu=3p$.

An example of a space-time with metric (4.1) which is not conformally flat, admits a CKV, and has T_{ab} of Segré type $\{2,(1\ 1)\}$, is

$$ds^2=-du^2-2\,du\,dv+P^{-2}(dx^2+dy^2), \tag{4.39}$$

with $P=u^n e^{x^2+y^2}$, where $-1<n<0$, $n\neq-\frac{1}{2}$. The CKV is

$$\zeta^a=u^{-2n}(-\delta_0^a+\tfrac{1}{2}\delta_1^a), \quad \psi=nu^{-2n-1}. \tag{4.40}$$

This can be interpreted as a viscous fluid model with

$$\begin{aligned} &u^a=\delta_0^a, \quad e_a=\delta_a^1, \quad \mu=4P^2-2n(n+1)u^{-2},\\ &p=-\tfrac{4}{3}P^2-\tfrac{2}{3}n(n+1)u^{-2},\\ &Q=-2n(n+1)u^{-2},\\ &\eta=P^2n^{-1}u+(n+1)u^{-1}. \end{aligned} \tag{4.41}$$

The conditions $\mu>0$, $Q>0$ always hold, while $p>0$, $\eta>0$ hold in the region of space-time defined by $2e^{2(x^2+y^2)}<-n(n+1)u^{-2n-2}$. Note that $n=-\frac{1}{2}$ is excluded because then $\psi=\text{const}$, i.e., ζ^a is a HV. The ACV is given by

$$\xi^a=-u^{-2n}\delta_0^a+\tfrac{1}{2}(u^{-2n}-u)\delta_1^a. \tag{4.42}$$

V. THE 2+2 CASE

When the space-time is locally decomposable into a $2+2$ space-time, Eq. (1.1) must be solved directly. The general $2+2$ metric can be written in the form

$$ds^2=-2e^{2q}\,du\,dv+2e^{2r}\,d\zeta\,d\bar{\zeta}, \tag{5.1}$$

where $q=q(u,v)$ and $r=r(\zeta,\bar{\zeta})$ are real-valued functions, u,v $(=x^0,x^1)$ are real null coordinates and $\zeta,\bar{\zeta}$ $(=x^2,x^3)$ are conjugate complex coordinates. The only possible form for the symmetric covariantly constant tensor K_{ab} for the metric (5.1) is

$$K_{01}=-he^{2q}, \quad K_{23}=ke^{2r}, \tag{5.2}$$

where h,k are constants with $h\neq k$ (otherwise, $K_{ab}=hg_{ab}$). Without loss of generality, we may put $k=0$.

The nonzero Ricci tensor components for the metric (5.1) are

$$R_{01}=-2q_{uv}, \quad R_{23}=-2r_{\zeta\bar{\zeta}}, \tag{5.3}$$

and the Ricci scalar is

$$R=4q_{uv}e^{-2q}-4r_{\zeta\bar{\zeta}}e^{-2r}. \tag{5.4}$$

The ACV equations (1.1) for the metric (5.1) with (5.2) are

$$\xi^0_{,v}=\xi^1_{,u}=\xi^2_{,\bar{\zeta}}=\xi^3_{,\zeta}=0, \tag{5.5}$$

$$e^{2q}\xi^0_{,\zeta}=e^{2r}\xi^3_{,v}, \tag{5.6}$$

$$e^{2q}\xi^0_{,\bar{\zeta}}=e^{2r}\xi^2_{,v}, \tag{5.7}$$

$$e^{2q}\xi^1_{,\zeta}=e^{2r}\xi^3_{,u}, \tag{5.8}$$

$$e^{2q}\xi^1_{,\bar{\zeta}}=e^{2r}\xi^2_{,u}, \tag{5.9}$$

$$\xi^0_{,u}+\xi^1_{,v}+2q_u\xi^0+2q_v\xi^1=2\psi+h, \tag{5.10}$$

$$\xi^2_{,\zeta}+\xi^3_{,\bar{\zeta}}+2r_\zeta\xi^2+2r_{\bar{\zeta}}\xi^3=2\psi. \tag{5.11}$$

Integrating Eq. (5.5) yields

$$\xi^a=[A(u,\zeta,\bar{\zeta}),B(v,\zeta,\bar{\zeta}),C(u,v,\zeta),\bar{C}(u,v,\bar{\zeta})] \tag{5.12}$$

and Eqs. (5.6)–(5.11) become

$$\begin{aligned} &e^{2q}A_\zeta=e^{2r}\bar{C}_v, \quad e^{2q}A_{\bar{\zeta}}=e^{2r}C_v,\\ &e^{2q}B_\zeta=e^{2r}\bar{C}_u, \quad e^{2q}B_{\bar{\zeta}}=e^{2r}C_u. \end{aligned} \tag{5.13}$$

$$(e^{2q}A)_u+(e^{2q}B)_v=e^{2q}(2\psi+h), \tag{5.14}$$

$$(e^{2r}C)_\zeta+(e^{2r}\bar{C})_{\bar{\zeta}}=e^{2r}2\psi. \tag{5.15}$$

For an ACV, ξ^a, the Lie derivative along ξ^a of the Ricci tensor and the Ricci scalar satisfy

$$\underset{\xi}{\mathscr{L}}R_{ab}=-2\psi_{;ab}-\Box\psi g_{ab}, \tag{5.16}$$

$$\mathop{\mathscr{L}}_{\xi} R = -2\psi R - 6\Box\psi - K^{ab}R_{ab}. \tag{5.17}$$

Setting $a=0$, $b=2$ in Eq. (5.16) yields

$$\mathop{\mathscr{L}}_{\xi} R_{02} = R_{32}\xi^3_{,u} + R_{01}\xi^1_{,\zeta} = -2\psi_{u\zeta};$$

i.e.,

$$\psi_{u\zeta} = r_{\zeta\bar{\zeta}}\bar{C}_u + q_{uv}B_\zeta.$$

Using Eq. (5.13) this becomes

$$\psi_{u\zeta} = Ke^{2q}B_\zeta = Ke^{2r}\bar{C}_u, \tag{5.18}$$

where K is defined by

$$K(u,v,\zeta,\bar{\zeta}) = q_{uv}e^{-2q} + r_{\zeta\bar{\zeta}}e^{-2r}. \tag{5.19}$$

Differentiating Eq. (5.15) wrt u using Eq. (5.13), we obtain

$$\psi_u = e^{2q-2r}B_{\zeta\bar{\zeta}}, \tag{5.20}$$

and differentiating this expression wrt ζ and using Eq. (5.18) yields

$$\psi_{u\zeta} = e^{2q}(e^{-2r}B_{\zeta\bar{\zeta}})_\zeta = Ke^{2q}B_\zeta, \tag{5.21}$$

from which it follows that either $K_u=0$ or $B_\zeta=0$. Similarly, by finding the mixed second partial derivatives $\psi_{u\bar{\zeta}}$, $\psi_{v\zeta}$, $\psi_{v\bar{\zeta}}$ in two ways and comparing the results, we find that either $K_u=K_v=K_\zeta=K_{\bar{\zeta}}=0$ or

$$A_\zeta = A_{\bar{\zeta}} = \beta_\zeta = B_{\bar{\zeta}} = C_u = C_v = \bar{C}_u = \bar{C}_v = 0. \tag{5.22}$$

However, if Eq. (5.22) holds, then Eqs. (5.14) and (5.15), respectively, show that $\psi=\psi(u,v)$ and $\psi=\psi(\zeta,\bar{\zeta})$; i.e., ψ is a constant so that ξ^a is not an ACV but an AV. Hence, K must be a constant and, from Eq. (5.19), this implies that $q_{uv}e^{-2q}$ and $r_{\zeta\bar{\zeta}}e^{-2r}$ are each constants, so that each two-space is a space of constant curvature. If κ_1 and κ_2 are, respectively, the constant curvature of the (u,v)-space and the $(\zeta,\bar{\zeta})$-space, then

$$\kappa_1 = 2q_{uv}e^{-2\mu}, \quad \kappa_2 = -2r_{\zeta\bar{\zeta}}e^{-2\nu}, \tag{5.23}$$

$$K = \tfrac{1}{2}(\kappa_1 - \kappa_2). \tag{5.24}$$

Differentiating the third equation of Eq. (5.13) wrt ζ and using Eq. (5.13), again we obtain

$$e^{2q}B_{\zeta\zeta} = 2r_\zeta e^{2r}\bar{C}_u = 2r_\zeta e^{2q}B_\zeta;$$

i.e.,

$$B_{\zeta\zeta} = 2r_\zeta B_\zeta;$$

i.e.,

$$B_{\zeta\zeta\bar{\zeta}} = 2r_\zeta B_{\zeta\bar{\zeta}} + 2r_{\zeta\bar{\zeta}}B_\zeta. \tag{5.25}$$

From Eqs. (5.21) and (5.25) we find

$$\psi_{u\zeta} = e^{2q-2r}(B_{\zeta\bar{\zeta}\zeta} - 2r_\zeta B_{\zeta\bar{\zeta}}) = e^{2q-2r}2r_{\zeta\bar{\zeta}}B_\zeta,$$

$$= \kappa_2 e^{2q}B_\zeta. \tag{5.26}$$

But, from Eqs. (5.18) and (5.24), we have

$$\psi_{u\zeta} = \tfrac{1}{2}(\kappa_1 - \kappa_2)e^{2q}B_\zeta, \tag{5.27}$$

whence comparison with Eq. (5.26) shows that

$$\kappa_1 + \kappa_2 = 0, \tag{5.28}$$

unless $B_\zeta=0$, which implies that $C_u=0$. If instead we use, for example, the first member of Eq. (5.13), we would again obtain Eq. (5.28) unless $A_\zeta=0$, which implies that $C_u=0$. Thus, using all of Eq. (5.13), we find that either Eq. (5.28) holds or that Eq. (5.22) holds; i.e., $\psi=$ const. Hence, Eq. (5.28) is always true for ACV and this is precisely the condition required for the space-time (5.1) to be conformally flat and, from (5.4) and (5.23), also implies that $R=0$. Thus the only possible space-times are the Bertotti–Robinson space-time, with $\kappa_1<0$, $\kappa_2>0$ and the "anti-Bertotti–Robinson" space-time with $\kappa_1>0$, $\kappa_2<0$. The second of these solutions does not satisfy any required energy conditions. Note that this result is independent of the value of the constant h and so holds for CKV; i.e., $h=0$.

Having found the space-time, we need to complete the investigation by finding the ACV ξ^a. We note that the Lie derivative along ξ^a of the Weyl tensor C^a_{bcd} is given by

$$\mathop{\mathscr{L}}_{\xi} C^a_{bcd} = \tfrac{1}{2}(g_{bd}K^{ae}R_{ec} - g_{bc}K^{ae}R_{ed} - K_{bd}R^a_c + K_{bc}R^a_d)$$
$$+ \tfrac{1}{6}(\delta^a_c K_{bd} - \delta^a_d K_{bc})R$$
$$- \tfrac{1}{6}(\delta^a_c g_{bd} - \delta^a_d g_{bc})K^{ef}R_{ef}. \tag{5.29}$$

Since for the space-time solutions found $C^a_{bcd}=0$, $R=0$, $K_{ab}=0$, except for $K_{01}=-he^{2q}$ and $R_{ab}=0$, except for $R_{01}=-\kappa_1 e^{2q}$, $R_{23}=\kappa_2 e^{2r}$, the (3,323) component of Eq. (5.29) is simply

$$\tfrac{1}{3}h\kappa_1 e^{2r} = 0,$$

and since $\kappa_1=0$ implies that $\kappa_2=0$, i.e., flat space-time, we conclude that $h=0$, so that no ACV exists for a nonflat $2+2$ space-time.

Turning to the case of a $1+1+2$ space-time, this is a special case of a $2+2$ space-time in which one two-space is flat, i.e., there exist two covariantly constant vector fields, $\eta_a \equiv \eta_{,a}$ and $\lambda_a \equiv \lambda_{,a}$, which may be both spacelike or one may be timelike and the other spacelike (or, equivalently, both may be null). Hence, in the metric (5.1), either $q=0$ or $r=0$. Excluding multiples of g_{ab}, the general form of K_{ab} in this case is

$$K_{ab}=\alpha\eta_{,a}\eta_{,b}+\beta(\eta_{,a}\lambda_{,b}+\eta_{,b}\lambda_{,a})+\gamma\lambda_{,a}\lambda_{,b},$$

where α, β, and γ are arbitrary constants, and Eq. (1.1) can be written in the form

$$(\xi_a-\tau_a)_{;b}+(\xi_b-\tau_b)_{;a}=2\psi g_{ab},$$

where

$$\tau_a=\tfrac{1}{2}\alpha\eta\eta_{,a}+\tfrac{1}{2}\beta(\eta\lambda_{,a}+\lambda\eta_{,a})+\tfrac{1}{2}\gamma\lambda\lambda_{,a}.$$

Hence, $\xi_a=\zeta_a+\tau_a$, where ζ_a is a CKV and τ_a is an AV [see Eqs. (1.3)–(1.6)]. Thus a $1+1+2$ space-time admitting an ACV must also admit a CKV. But we have shown that the only $2+2$ space-times admitting CKV are those that are conformally flat and, since 1 two-space is flat, it follows that the $1+1+2$ space-time must be flat Minkowski space-time. Hence, apart from this trivial case, $1+1+2$ space-times do not admit CKV and so do not admit ACV.

The results of this section may be summarized in the following theorem.

Theorem 2. A $2+2$ space-time cannot admit an ACV; this result holds true even if the space-time decomposes into a $1+1+2$ space-time. Furthermore, the only $2+2$ space-times admitting proper CKV are those which are conformally flat and, of these solutions, the Bertotti–Robinson solution is the only one that satisfies the energy conditions.

VI. CONCLUSION

We have shown that the only space-times admitting proper ACV are those with metrics (2.9)–(2.12), (3.9), (3.10), (4.1) satisfying (4.26), but not (4.29), and (4.3) satisfying (4.34)–(4.37). Of these, only (2.9), (3.11), and some of (4.1) and (4.3) satisfy the dominant energy condition. In addition, there is the trivial case of flat Minkowski space-time that admits four proper CKV and four covariantly constant vectors and so admits many ACV given by the sum of any CKV with various combinations of the covariantly constant vectors.

The ACV space-times are not only few in number, they also admit very restrictive physical interpretations. The static Einstein universe is the only perfect fluid space-time, there are no non-null Einstein–Maxwell space-times, and the only null Einstein–Maxwell space-time is the conformally flat *pp*-wave solution. Otherwise, the ACV space-times can represent anisotropic fluids or viscous heat-conducting fluids; in the former case, the equations of state are very restrictive, while in the latter case they are usually quite restrictive. Thus the solution to the mathematical problem of finding all space-times admitting ACV shows that these space-times are physically less interesting than envisioned by previous authors.

Levine and Katzin[4] have shown that if a nonflat conformally flat space admits a covariantly constant symmetric second-rank tensor L_{ab}, then the two fundamental solutions for L_{ab} are $L_{ab}=\text{const}\times g_{ab}$ and $L_{ab}=\text{scalar}\times R_{ab}$, i.e., in general

$$L_{ab}=Vg_{ab}+WR_{ab}, \tag{6.1}$$

where V is a constant and W is a scalar function. Identifying L_{ab} with $K_{ab}=\eta_{,a}\eta_{,b}$, it is clear that V and W are determined by Eq. (1.1). The only two conformally flat space-times admitting ACV are the Einstein static universe and the special case of the *pp*-wave space-time in which the metric function H is given by

$$H=f(u)(x^2+y^2), \tag{6.2}$$

where $f(u)$ is an arbitrary function of u. For the Einstein universe we find that $V=-1$, $W=\frac{1}{2}$; i.e.,

$$K_{ab}=\eta_{,a}\eta_{,b}=-g_{ab}+\tfrac{1}{2}R_{ab}. \tag{6.3}$$

Note that we cannot include the g_{ab} term on the left-hand side, i.e., we cannot write

$$K_{ab}\equiv\eta_{,a}\eta_{,b}+g_{ab}=\tfrac{1}{2}R_{ab}, \tag{6.4}$$

because this would imply the existence of a HV (see Sec. I), which is not true of the Einstein universe. This shows that for a nonflat conformally flat space-time, K_{ab} is not necessarily proportional to R_{ab}, contrary to an assertion made in Ref. 8. For the conformally flat *pp*-wave case, we find that $V=0$, $W=\frac{1}{4}f^{-1}(u)$ so that in this case K_{ab} is proportional to R_{ab}.

Finally, we note that as a by-product of our investigation, we have shown that the only $2+2$ decomposable space-times admitting a proper CKV are those that are conformally flat and, of these, only the Bertotti–Robinson space-time satisfies the energy conditions.

ACKNOWLEDGMENTS

We are grateful to Clive Kilmister for providing the proof of Sinyukov's result in the Appendix and for his considerable help with the proof of the subsequent result. We are also grateful to Graham Hall for valuable comments and discussions. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada through operating grants to each author.

APPENDIX

Petrov[13] quotes without proof a result due to Sinyukov,[14] namely, if an n-dimensional Riemannian space of arbitrary signature, with metric tensor g_{ab}, admits a gradient vector field $\phi_{,a}$ satisfying

$$\phi_{;ab}=\rho g_{ab}, \tag{A1}$$

where ρ is a nonzero scalar function of the coordinates, then a system of coordinates exists in which the metric takes the form

$$ds_n^2=g_{11}(dx^1)^2+(g_{11})^{-1}\Gamma_{pg}(x^2,...,x^n)dx^p\,dx^q, \tag{A2}$$

where

$$p,q\neq 1,\quad g_{11}=\left[2\int\rho(x^1)dx^1+C\right]^{-1},$$

and ρ is now an arbitrary function of x^1 only. Since the proof of this result is nontrivial, we shall present a proof here, which is as follows.

Choose coordinates such that $\phi=x^1$ and choose the remaining coordinates x^p $(p\neq 1)$ to lie in the hypersurface $\phi=$ const. This implies that the metric tensor components $g_{1p}=0$, $\phi_{,a}=\delta^1_a$, and $g^{11}=(g_{11})^{-1}$. The components of Eq. (A1) are then

$$\phi_{;11}=\phi_{,11}-\left\{{a \atop 11}\right\}\phi_{,a}=-\left\{{1 \atop 11}\right\}=-\tfrac{1}{2}g^{11}g_{11,1}=\rho g_{11}, \tag{A3}$$

$$\phi_{;1p}=\phi_{,1p}-\left\{{1 \atop 1p}\right\}\phi_{,1}=-\left\{{1 \atop 1p}\right\}=-\tfrac{1}{2}g^{11}g_{11,p}=0, \tag{A4}$$

$$\phi_{;pq}=\phi_{,pq}-\left\{{1 \atop pq}\right\}\phi_{,1}=-\left\{{1 \atop pq}\right\}=\tfrac{1}{2}g^{11}g_{pq,1}=\rho g_{pq}. \tag{A5}$$

Equation (A4) implies that $g_{11}=g_{11}(x^1)$, while Eq. (A3) shows that

$$-\tfrac{1}{2}(g_{11})^{-2}g_{11,1}=\rho \tag{A6}$$

so that $\rho=\rho(x^1)$ and, on integration, we obtain

$$(g_{11})^{-1}=2\int\rho(x^1)dx^1+C. \tag{A7}$$

Equation (A5) then becomes

$$g_{11}g_{pq,1}+g_{11,1}g_{pq}=0;$$

i.e.,

$$g_{pq}=(g_{11})^{-1}\Gamma_{pq}(x^2,...,x^n), \tag{A8}$$

thus proving the result.

In Secs. II and III we are required to find space-times of the form

$$ds^2=\epsilon(dx^1)^2+g_{pq}(x^r)dx^p\,dx^q\quad(\epsilon=\pm 1) \tag{A9}$$

which admit a gradient vector field $\psi_{,a}$ satisfying

$$\psi_{;pq}=-\epsilon\psi_{,11}g_{pq}, \tag{A10}$$

where $p,q\neq 1$. This differs from Eq. (A1) in that, for a space-time, g_{pq} is strictly three dimensional, whereas ψ can depend on all four coordinates. Unlike the previous case we cannot, in general, choose ψ as a function of x^1 only, but we can choose ψ to be a function of x^1 and one other coordinate, say x^2, i.e., in each $x^1=$ const hypersurface we can choose the coordinate x^2 so that ψ is a function of x^2 only. We also choose the remaining coordinates x^A $(A=3,4)$ to lie in the $x^2=$ const hypersurfaces which implies that $g_{2A}=0$. Equation (A10) then has the following components:

$$\psi_{;22}=\psi_{,22}-\tfrac{1}{2}g^{22}g_{22,2}\psi_{,2}=-\epsilon\psi_{,11}g_{22}, \tag{A11}$$

$$\psi_{;2A}=-\tfrac{1}{2}g^{22}g_{22,A}\psi_{,2}=0, \tag{A12}$$

$$\psi_{;AB}=\tfrac{1}{2}g^{22}g_{AB,2}\psi_{,2}=-\epsilon\psi_{,11}g_{AB}. \tag{A13}$$

Now $\psi_{,2}\neq 0$; otherwise, Eq. (A11) implies that $\psi_{,11}=0$ and thus $\psi_{,pq}=0$ so that $\psi_{;ab}=0$ and the ACV becomes a SACV. Hence, Eq. (A12) implies that $g_{22}=g_{22}(x^2)$. From Eqs. (A11) and (A13), using the fact that $g^{22}=(g_{22})^{-1}$, we obtain

$$g_{AB,2}=(2\psi_{,22}/\psi_{,2}-g_{22,2}/g_{22})g_{AB}, \tag{A14}$$

which shows that $\psi_{,22}/\psi_{,2}$ is independent of x^1; i.e.,

$$\psi=a(x^2)b(x^1)+c(x^1). \tag{A15}$$

Equation (A14) then becomes

$$g_{AB,2}=(2a_{22}/a_{,2}-g_{22,2}/g_{22})g_{AB}, \tag{A16}$$

which integrates to give

$$g_{AB}=(a_{,2})^2(g_{22})^{-1}p_{AB}(x^C). \tag{A17}$$

Putting $g_{22}=M(x^2)$ and defining a new coordinate $x^{2'}$ by

$$M^{1/2}\,dx^2=dx^{2'}, \tag{A18}$$

the metric (A9) becomes, on dropping the primes,

$$ds^2=\epsilon(dx^1)^2+e(dx^2)^2+(a_{,2})^2p_{AB}(x^C)dx^A\,dx^B, \tag{A19}$$

where $e=\pm 1$ and at least one of ϵ,e takes the value $+1$. From Eqs. (A11) and (A15), we obtain

$$a_{,22}b=ab_{,11}+c_{,11} \tag{A20}$$

and differentiation wrt x^2 yields

$$a_{,222}/a_{,2} = b_{,11}/b = \chi,\text{const.} \tag{A21}$$

If $\chi \neq 0$, the first of these equations integrates to give, after some rescaling and translation of the coordinates

$$a_{,22} = \chi a, \tag{A22}$$

whereas, when $\chi = 0$, we obtain

$$a_{,22} = \text{const.} \tag{A23}$$

After further coordinate manipulations, the solutions of Eqs. (A22) and (A23) are

$$a_{,2} = \sin kx^2 \quad (\chi = -k^2), \tag{A24}$$

$$a_{,2} = \cosh kx^2 \quad \text{or} \quad a_{,2} = \sinh kx^2 \quad (\chi = k^2), \tag{A25}$$

$$a_{,2} = x^2 \quad (\chi = 0), \tag{A26}$$

and, with suitable scaling, b takes the values

$$b = -\cos kx^1 \quad (\chi = -k^2), \tag{A27}$$

$$b = \cosh kx^1 \quad \text{or} \quad b = \sinh kx^1 \quad (\chi = k^2), \tag{A28}$$

$$b = \beta x^1 \quad (\chi = 0), \tag{A29}$$

where β is an arbitrary constant.

From Eq. (A20), $C_{,11} = 0$, when $\chi = \pm k^2$; i.e.,

$$C = \gamma x^1 + \delta, \tag{A30}$$

but, when $\chi = 0$, $C_{,11} = b$; i.e.,

$$C = \tfrac{1}{6}\beta(x^1)^3 + \sigma x^1 + \omega, \tag{A31}$$

where γ, δ, σ, ω are arbitrary constants. In all cases ψ is given by Eq. (A15).

Thus the space-times of the form (A9), which admit a gradient vector field $\psi_{,a}$ satisfying Eq. (A10), are locally of the form (A19) with $a_{,2}$ given by one of the four expressions (A33)–(A35). The corresponding scalar ψ is given by Eq. (A15), with a being the integral of the appropriate a_2, b is given by one of the expressions (A36)–(A38) [note that either expression for b in Eq. (A37) can be associated with either expression for a obtained from Eq. (A34)], and c is given by the appropriate expression (A30) or (A31).

[1] A. A. Coley and B. O. J. Tupper, J. Math. Phys. **30**, 2616 (1989).
[2] A. A. Coley and B. O. J. Tupper, J. Math. Phys. **31**, 649 (1990).
[3] G. S. Hall and J. da Costa, J. Math. Phys. **29**, 2465 (1988).
[4] J. Levine and G. H. Katzin, Tensor (NS) **19**, 317 (1968).
[5] G. H. Katzin, J. Levine, and W. R. Davis, Tensor (NS) **21**, 51 (1970).
[6] D. P. Mason and R. Maartens, J. Math. Phys. **28**, 2182 (1987).
[7] K. L. Duggal, J. Math. Phys. **28**, 2700 (1987).
[8] K. L. Duggal, J. Math. Phys. **30**, 1316 (1989).
[9] Y. Tashiro, Math. J. Okayama Univ. **10**, 75 (1960).
[10] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge U.P., Cambridge, 1980).
[11] R. Maartens and S. D. Maharaj, Class. Quantum Grav. **8**, 503 (1991).
[12] A. A. Coley and B. O. J. Tupper, Gen. Relat. Gravit. **21**, 1113 (1991).
[13] A. Z. Petrov, *Einstein Spaces* (Pergamon, Oxford, 1969).
[14] N. S. Sinyukov, Scientific Annual, Odessa University (1957).