Averaging spherically symmetric spacetimes in general relativity

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We discuss the averaging problem in general relativity, using the form of the macroscopic gravity equations in the case of spherical symmetry in volume preserving coordinates. In particular, we calculate the form of the correlation tensor under some reasonable assumptions on the form for the inhomogeneous gravitational field and matter distribution. On cosmological scales, the correlation tensor in a Friedmann-Lemaître-Robertson-Walker (FLRW) background is found to be of the form of a spatial curvature. On astrophysical scales the correlation tensor can be interpreted as the sum of a spatial curvature and an anisotropic fluid. We briefly discuss the physical implications of these results.

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The gravitational field equations on large scales are obtained by averaging the Einstein equations of general relativity (GR). The Universe is not isotropic or spatially homogeneous on local scales. An averaging of inhomogeneous spacetimes on large scales can lead to important effects. For example, on cosmological scales the dynamical behavior can differ from that in the spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) model [1]; in particular, the expansion rate may be significantly affected. Consequently, a solution of the averaging problem is of considerable importance for the correct interpretation of cosmological data. It is also of importance for physical phenomena on astrophysical (galactic) scales.

There are a number of theoretical approaches to the averaging problem [2–4]. In the approach of Buchert [4] a $3 + 1$ cosmological spacetime splitting is employed and only scalar quantities are averaged. The perturbative approach [2] involves averaging the perturbed Einstein equations; however, a perturbation analysis alone cannot provide detailed information about an averaged geometry. On the other hand, the macroscopic gravity (MG) approach to the averaging problem in GR [3] gives a prescription for the correlation functions which emerge in an averaging of field equations. The MG approach is consequently a fully covariant, gauge independent and exact method. In particular, in the MG approach tensor fields are averaged over 4-volume averaging regions (rather than spatial averaging regions) about points of the microscopic spacetime. The resulting object is extended to a field defined over the macroscopic spacetime by Lie-dragging of 4-volume averaging regions. By employing a bilocal operator acting on tensor fields of the microscopic spacetime, the now modified averaging operator guarantees tensorial objects (i.e., covariance) in the macroscopic spacetime. We shall adopt the MG averaging approach. Averaging of the structure equations for the geometry of GR then leads to the structure equations for the averaged (macroscopic) geometry and the definitions and the properties of the correlation tensor. The averaged Einstein equations can always be written in the form of the Einstein equations for the macroscopic metric tensor when the correlation terms are moved to the right-hand side of the averaged Einstein equations [3].

Spherical symmetry is of particular physical interest, and it is especially important to study the averaging problem within the class of spherically symmetric cosmological models. In [5] the microscopic field equations were taken and the averaging procedure was effected to determine the precise form of the correlation tensor in this case. In volume preserving coordinates (VPC), the spherically symmetric line element is given by

$$ds^2 = -Bdt^2 + Adr^2 + \frac{du^2}{\sqrt{AB}(1-u^2)} + \frac{1-u^2}{\sqrt{AB}}d\phi^2, \tag{1}$$

where the functions A and B depend on t and r. The FLRW metric in VPC is given by (1), with $A = R^2/F^4$, $B = 1/R^6$, where $R = R(t)$ and $F = F(r)$, subject to $\frac{dF}{dr} = \sqrt{1-kF^2}/F^2$ and $k = -1$, 0 or 1. We can calculate the form of the Einstein tensor $G^a{}_b$, take averages, and obtain the appropriate form for the MG field equations and hence the correlation tensor $C^a{}_b$ (for example, we have that $C^r{}_t = G^r{}_t\langle g\rangle - \langle G^r{}_t\rangle$). In VPC, in which the bilocal operator acts as an identity operator, the average is then simply given by

$$\langle f(r,t)\rangle \equiv \frac{1}{TL}\int_{t'=-(T/2)}^{T/2} dt' \int_{r'=-(L/2)}^{L/2} dr' f(r+r', t+t'), \tag{2}$$

which, for smooth functions with a slowly varying dependence on cosmological time, essentially reduces to a spatial average in terms of the averaging scale L (with $L \equiv h_0/H < 1$).

The form of the correlation tensor depends on the assumed form for the inhomogeneous gravitational field and matter distribution (and may depend on the choice of VPC [5]). We assume that

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$$A(r,t) = \langle A(r,t)\rangle\left[1 + \sum_{n=1}^{\infty} a_n(t)L^n \sin\left(\frac{2n\pi}{L}r\right) + \sum_{n=1}^{\infty} \bar{a}_n(t)L^n \cos\left(\frac{2n\pi}{L}r\right)\right], \quad (3)$$

$$B(r,t) = \langle B(r,t)\rangle\left[1 + \sum_{n=1}^{\infty} b_n(t)L^n \sin\left(\frac{2n\pi}{L}r\right) + \sum_{n=1}^{\infty} \bar{b}_n(t)L^n \cos\left(\frac{2n\pi}{L}r\right)\right], \quad (4)$$

where the inhomogeneous functions A and B satisfy a set of appropriate and self-consistent conditions (for example, $\langle\frac{\partial}{\partial t}\langle A(r,t)\rangle\rangle = \langle\frac{\partial A(r,t)}{\partial t}\rangle$). The assumptions (3) and (4) constitute a spatial Fourier decomposition of the metric functions in which the variation in the timelike direction is assumed small and the dominant source of inhomogeneity arises from a spatial variation of the gravitational field (thus the 4-volume average effectively reduces, in this case, to a smoothing on a spatial domain). Note that the coordinates t and r appearing in (1) are not the usual "time" and "radial" coordinates; however, the unit magnitude timelike coordinate basis vector has zero vorticity, which implies the existence of a foliation of spacetime (where the r coordinate parameterizes the spatial hypersurfaces). Since the coordinate basis vectors ∂_t and ∂_r are independent (i.e., the metric is diagonal), it follows that variation along timelike and spatial directions is not coupled. Although other forms for the inhomogeneous gravitational field are possible (i.e., different assumptions to (3) and (4)), it is not expected that the main conclusions here will be affected (see, for example, [5]).

Expanding in powers of $L < 1$, we obtain the correlation tensor up to $\mathcal{O}(L^2)$ [5]:

$$C^a{}_b = \mathrm{diag}\left[C + \frac{2\ell}{\langle A\rangle}, C, \frac{\ell}{\langle A\rangle}, \frac{\ell}{\langle A\rangle}\right], \quad (5)$$

where $C \equiv C^r{}_r$ and

$$\ell(t) \equiv \frac{\pi^2}{8}[(a_1 - 3b_1)(a_1 + b_1) + (\bar{a}_1 - 3\bar{b}_1)(\bar{a}_1 + \bar{b}_1)].$$

The function C can then be calculated from the contracted Bianchi identities. We note that if $C^a{}_b$ is isotropic (i.e., of the form of a perfect fluid) then $C = \frac{\ell}{\langle A\rangle}$ and $C^a{}_b$ is of the form of a spatial curvature term. Hereafter, for convenience we shall drop the angled brackets on averaged quantities.

Let us first discuss averaging on cosmological scales. In the case that $B_r = 0$, as in the case of a FLRW background, the contracted Bianchi identities immediately yield $C \equiv \ell/A$ and $A_r = 0$, and $\ell/A = \ell_0 R^{-2}$ (where ℓ_0 is a constant). Therefore, in this case we obtain

$$C^a{}_b = \ell_0 R^{-2}\mathrm{diag}[3, 1, 1, 1], \quad (6)$$

and $C^a{}_b$ is necessarily of the form of a spatial curvature term.

The cosmological result that in the spherically symmetric case the averaged Einstein equations in an FLRW background have the form of the Einstein equations of GR for a spatially homogeneous, isotropic macroscopic spacetime geometry with an additional spatial curvature term, confirms the results in previous work in which we were able to explicitly solve the MG equations to find a correction term (correlation tensor) in the form of a spatial curvature [6]. This result is also (i) consistent with the work of Buchert [4], in which a spatial curvature term appears when averaging in a FLRW background, (ii) consistent with the results of averaging an exact Lemaître-Tolman-Bondi (LTB) spherically symmetric dust model [7], in which solutions of the LTB metric in (nondiagonal) VPC are given explicitly as perturbations about the spatially flat FLRW model and found to give rise to solutions which can be interpreted as having both spatial curvature and a constant correction term, and (iii) consistent with results in which the effects of linear inhomogeneous perturbations on an exact spatially homogeneous and isotropic FLRW background [2,8] are found to give rise to correlation terms of the form of a spatial curvature term.

Inhomogeneities can affect the dynamics and may significantly affect the expansion rate of the spatially averaged "background" FLRW universe (the effect depending on the scale of the initial inhomogeneity) [2]. Therefore, a more conservative approach to explain the acceleration of the Universe [9] without introduction of exotic fields is to utilize a back-reaction effect due to inhomogeneities of the Universe. Indeed, it has been suggested that back-reactions from inhomogeneities could explain the apparently observed accelerated expansion of the niverse today. This has been investigated by studying the effective Friedmann equation describing an inhomogeneous Universe after averaging, using both perturbative and qualitative analyses [8,10]. It is clear that the perturbative effect proposed always gives rise to a renormalisation of the spatial curvature. It has also been argued that the effect does not simply reduce to spatial curvature and an acceleration can also result (although it is unlikely to be compatible with other observational data).

The MG method adopted here is an exact approach in which inhomogeneities affect the dynamics on large scales through the correlation term (and hence the main criticisms of the back-reaction approach to studying the possible contributions to an accelerated expansion [8,10] do not apply here). Averaging can have a very significant dynamical effect on the evolution of the Universe; the correction terms change the interpretation of observations so that they need to be accounted for carefully to determine if the models may be consistent with an accelerating Universe. Averaging may or may not explain the observed acceleration. However, it is clear that it cannot be neglected, and a proper analysis will not be possible without a comprehensive understanding of the affects of averaging.

Let us next consider the effects of averaging on astrophysical scales (e.g., galactic scales). We assume that a galaxy can be approximated as spherically symmetric. In a non-FLRW background (with $B_r \neq 0$), the contracted Bianchi identities can then be integrated to obtain [5]

$$C = -\frac{\ell}{A} + f(t)\frac{(AB)^{1/2}}{A^{2\ell}}; \qquad \dot{\ell} = -\left[\frac{2f}{A^{2\ell}}\right]_{,t} A^{3/2}B^{1/2}. \tag{7}$$

We note that $C^a{}_b$ is necessarily anisotropic (and cannot be formally equivalent to a perfect fluid). For the solution with $\ell = \ell_0 =$ const. and $2f = g(r)A^{2\ell}$, we can always write

$$C^a{}_b = \ell_0 A^{-1}\mathrm{diag}[3, 1, 1, 1] - \Pi\,\mathrm{diag}[1, 1, 0, 0] \tag{8}$$

where $\Pi \equiv -\{g(r)AB^{1/2} - 2\ell_0 A^{-1}\}$. The correlation tensor $C^a{}_b$ then automatically satisfies the contracted Bianchi identities. It can be interpreted as the sum of a perfect fluid and an anisotropic fluid (when $B_r \neq 0$). If both terms separately satisfy the contracted Bianchi identities, then the first term can be interpreted as a spatial curvature term and the second term can be interpreted as an anisotropic fluid with $p_\perp = 0$ and $p_\parallel = -\rho_{\mathrm{eff}}$. For an anisotropic fluid in spherically symmetric models the energy-momentum tensor is of the form $\mathrm{diag}[-\mu, p_\parallel, p_\perp, p_\perp]$, where $p_\parallel = p + \frac{2}{3}\pi$ and $p_\perp = p - \frac{1}{3}\pi$, and π is the anisotropic pressure. From above, we see that if the (total) correlation tensor $C^a{}_b$ is interpreted as an anisotropic fluid (which is comoving in VPC), it follows that $\Pi = -\pi$ and $p = -\frac{1}{3}\mu$. Anisotropic fluids in spherically symmetric models have been studied in [11].

Although the correlation tensor $C^a{}_b$ satisfies the contracted Bianchi identities, when interpreted as the sum of a spatial curvature perfect fluid and an anisotropic fluid through (8), the two separate fluids do not in general satisfy separate conservation equations. However, the contracted Bianchi identities can be rewritten in the form of a conservation law for the anisotropic pressure π,

$$\pi_t - \frac{1}{2}\pi\left(\frac{A_t}{A} - \frac{B_t}{B}\right) + \frac{\ell_0}{A}\left(2\frac{A_t}{A} + \frac{B_t}{B}\right) = 0, \tag{9}$$

in VPC where the metric is given by Eq. (1).

Let us comment on the astrophysical applications of an anisotropic fluid. It is known that dark matter is a major constituent of the halos of galaxies [12]. By an analysis of observed rotation curves, under reasonable assumptions (e.g., that galaxies can be modeled as spherically symmetric) it has been found that the dark matter is of the form of an anisotropic fluid [13]. This has been taken up in [14], in which the consequences of anisotropic dark matter stresses are discussed in weak field gravitational lensing (where it was noted that any attempt to model dark matter in galactic halos with classical fields will lead to anisotropic stresses comparable in magnitude with the energy density).

It is of interest to further study the effects of averaging in the astrophysical context. The results of this work could be used to model the effects phenomenologically by including an anisotropy term (comoving in VPC), which in general has $p = -\frac{1}{3}\mu$, and in the case $\ell_0 = 0$ is of the specific form $p_\perp = 0$ and $p_\parallel = -\rho_{\mathrm{eff}}$ (where the correlation tensor is given by $-\Pi\,\mathrm{diag}[1, 1, 0, 0]$). The anisotropic fluid satisfies the Bianchi identities, but since it arises from an averaging procedure it need not satisfy any energy conditions. It may be beneficial to work in VPC, in which the metric is diagonal and the correlation tensor is "comoving" (although the matter is not generally comoving). Indeed, in VPC the correlation tensor is given explicitly in terms of the averaged metric functions (e.g., $\mu = -g(r)AB^{1/2} - \ell_0 A^{-1}$, $p = -\frac{1}{3}\mu$, $\pi = g(r)AB^{1/2} - 2\ell_0 A^{-1}$). A disadvantage is that astrophysicists are not familiar with working in these coordinates. Alternatively, we could transform back to more conventional coordinates and determine the form of the correlation tensor; however, these coordinates may not be the most natural (e.g., the metric will not be diagonal) and the form of the correlation tensor (which is no longer comoving) may be quite complicated.

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